Perturbation Theory from Automorphic Forms

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Abstract

Using our previous construction of Eisenstein-like automorphic forms we derive formulae for the perturbative and non-perturbative parts for any group and representation. The result is written in terms of the weights of the representation and the derivation is largely group theoretical. Specialising to the E_{n+1} groups relevant to type II string theory and the representation associated with node $n + 1$ of the E_{n+1} Dynkin diagram we explicitly find the perturbative part in terms of String Theory variables, such as the string coupling g_d and volume V_n. For dimensions seven and higher we find that the perturbation theory involves only two terms. In six dimensions we construct the $SO(5,5)$ automorphic form using the vector representation. Although these automorphic forms are generally compatible with String Theory, the one relevant to R^4 involves terms with g_d^{-6} and so is problematic. We then study a constrained $SO(5,5)$ automorphic form, obtained by summing over null vectors, and compute its perturbative part. We find that it is consistent with String Theory and makes precise predictions for the perturbative results. We also study the unconstrained automorphic forms for E_6 in the **27** representation and E_7 in the **133** representation, giving their perturbative part and commenting on their role in String Theory.

1. Introduction

The low energy effective actions for the type II superstring theories are the IIA [1,2,3] and IIB [4,5,6] supergravity theories. They are complete in that they contain all perturbative and non-perturbative effects and are essentially unique. To date there does not exist a non-peturbative formulation of String Theory and these supergravity theories, as well as the supergravity theory in eleven dimensions [7], have been essential for our present understanding.

The maximal supergravity theories in dimensions less than ten can be obtained from the eleven or ten dimensional theories by dimensional reduction. One of the most unexpected developments was the realisation that these theories possess exceptional symmetries. If one dimensionally reduced the IIB theory in ten dimensions on an n torus then the resulting theory has an E_{n+1} symmetry. In particular one finds an E_6 symmetry in five dimensions, E_7 symmetry in four dimensions [8], E_8 in three dimensions [9] and E_9 in two dimensions [10,11,12]. In six, seven and eight dimensions we find the groups $E_5 = SO(5,5)$, $E_4 = SL(5)$ and $E_3 = SL(2) \times SL(3)$ respectively [13]. While the IIB supergravity theory in ten dimensions has an $E_1 = SL(2)$ symmetry [4].

All these supergravity theories possess solitonic solutions whose charges obey quantisation conditions [14,15] and as the E_{n+1} symmetry groups act on the charges only a discrete subgroup can persist in the quantum theory, instead of the continuous symmetries found in the supergravity theories. Such phenomena were first pointed out in a supergravity context for the four-dimensional heterotic string [16], which had already been conjectured to possess an S duality [17]. For type II String Theories in all dimensions it was conjectured that a discrete E_{n+1} symmetry, well-known as U-duality, were symmetries of the full quantum theory [18].

Despite the intense interest in U-duality there has only been limited work testing this conjecture beyond the low energy, *i.e.* supergravity, approximation. The most notable exception has been the ten dimensional IIB string theory. In particular, it was proposed that the coefficients of the R^4 and $D^2 R^4$ and certain other higher order terms were specific non-holomorphic automorphic forms for $SL(2, Z)$ of Eisenstein type [19-25]. These objects were analysed and it was found that they predicted all the perturbative and non-perturbative behaviour associated with these corrections. Indeed these authors were able to show that there was considerable agreement between these predictions at the perturbative and non-perturbative level and the ones of IIB string theory. Terms of the form $R^4 H^{4g-4}$ were also considered and found to have similar interesting features [26].

The evidence for U-duality in lower dimensions is less strong. Automorphic forms in eight dimensions for the group $SL(2) \otimes SL(3)$ [27,28], and also for seven dimensions for the group $SL(5)$ [27], have been considered and some agreement with string theory predictions has been found. A general discussion of automorphic forms for $SL(N)$ and $SO(d,d)$

was given in reference [29] which derived some of their properties and considered their connection to string theory, in particular the relation between BPS states and constrained automorphic forms.

A slightly different approach was taken in references [30-33] which directly looked for evidence for E_{n+1} symmetries in the higher derivative corrections. One of the simplest ways to find strong evidence for the existence of the E_{n+1} symmetries in the supergravity theories is to compute the dependence of the theory on the diagonal components of the metric $\vec{\phi}$ associated with the n-torus. One finds terms with factors of $e^{\sqrt{2}\vec{w}\cdot\phi}$ where the vectors $\vec{w}$ are the roots of E_{n+1}. Thus although not a proof, one clearly sees the E_{n+1} Lie algebra emerge in a very transparent way. The same calculation for the higher derivative corrections does not lead to roots but rather the weights of E_{n+1} [30,31]. Reference [31] also gave a general construction of non-holomorphic automorphic forms for any group G, based on the theory on non-linear realisations, including those that transformed non-trivially. An automorphic form was constructed from a given representation of G and it was shown that it involved weights of G. Thus the appearance of weights of E_{n+1} in the dimensional reduction was evidence for the appearance of non-holomorphic automorphic forms. Furthermore the highest weight of the representation on which the automorphic form was based could be deduced from the dimensional reduction. Thus the dimensional reduction of the higher derivative corrections provides evidence for the appearance of automorphic forms and so for an E_{n+1} symmetry in the higher derivative corrections [31].

Demanding a discrete E_{n+1} invariance of the complete string theory effective action implies that the coefficients of the higher derivative terms transform in a specific way, which is the same as the automorphic forms that are well studied in the mathematical literature. The class of functions described in reference [31], and which are used in this paper, are constructed to have the correct transformation property suitable for any higher derivative term. However, it was not demanded that these functions should be eigenfunctions of the Laplacian or other Casimir operators, unlike the automorphic forms in the mathematical literature. While it is known that the coefficients of certain higher derivative terms that have low numbers of spacetime derivatives are eigenfunctions of the Laplacian [21] this is not the case in general (e.g. see [23]). Therefore we will use the term automorphic form to be any function with the appropriate transformation property under E_{n+1} and not impose any constraints on Casimir operators.

We will also consider automorphic forms constructed as in [31] but whose lattice sums are subject to E_{n+1}-invariant constraints. In certain cases imposing quadratic constraints does result in an eigenfunction of the Laplacian [29]. However, as already pointed out in [31], the automorphic forms that occur, at least for low numbers of spacetime derivatives, are likely to also be eigenfunctions of the higher order Casimir operators, and these can presumably be obtained by imposing higher order E_{n+1}-invariant constraints.

It has become clear that higher derivative string corrections in $d = 10 - n$ dimensions are controlled by non-holomorphic automorphic forms of E_{n+1} and these contain all the perturbative and non-perturbative effects. One of the most striking features to emerge was that there were very few perturbative corrections implying novel non-renormalisaton theorems, some which have been verified in String Theory [34] and state that certain operators only receive contributions from two orders of perturbation theory. However, the relatively small number of papers on this subject is a testament to the difficulties of working with non-holomorphic automorphic forms, some recent studies include [35],[36].

This paper is organised as follows. In section 2 we will use our previous method [31] to construct Eisenstein-like automorphic forms for any group representation and in section 3 find explicit formulae for the "perturbative" and "non-perturbative" parts. Since our construction is based on representation theory the derivation of these formulae is largely group theoretic and the result is expressed in terms of the weights of the representation being used to construct the automorphic form. In section 4 we will discuss the analytic continuation and regulation of these forms that is required to define the automorphic forms that are expected to be relevant in String Theory. In section 6 we then apply the perturbative formula to the groups E_{n+1} and the fundamental representation associated with node $n + 1$ of the E_{n+1} Dynkin diagram (see figure 1), finding an explicit form of the perturbative part using the decomposition to $SL(2) \otimes SL(n)$. In section 7 we will compute the perturbative part in terms of the string quantities, that is the string coupling g_d in d dimensions and the volume of the torus V_n.

We will find that for the case of dimensions $d \geq 7$, that is automorphic forms for the groups $SL(2)$, $SL(2) \otimes SL(3)$ and $SL(5)$, that there is a physically acceptable perturbative series with contributions at only two orders. Acceptable here simply means it is consists of terms of the form g_d^{2g-2}. This a necessary condition for the automorphic form to appear in String Theory but it is certainly not a sufficient condition. However, this is not always the case for six dimensions, that is for the automorphic forms based on the group $SO(5,5)$ and the vector, *i.e.* the **10**, representation. In dimensions five and four we consider the automorphic forms for the exceptional groups E_6, with representation **27**, and E_7, with representation **133**, respectively. We discuss to what extend these perturbation series are consistent with string theory. In section 8 we then consider constrained $SO(5,5)$ automorphic forms based on a null vector and find that one always has an acceptable perturbation theory result which is explicitly derived in terms of string theory variables. Section 9 contains a discussion of our results.

Note Added: While this paper was in preparation we received [37] which contains some overlap of our work, in particular section 8. In addition reference [29] was recently revised and shortly after this paper appear we received [38] which also contains related results.

2. Construction of Automorphic Forms for G/H

Let us begin by a review of the construction of non-linear realisations. We consider a group G with Lie algebra $Lie(G)$. $Lie(G)$ can be split into the Cartan subalgebra with elements $\vec{H}$, positive root generators $E_{\vec{\alpha}}$ and negative root generators $E_{-\vec{\alpha}}$ with $\vec{\alpha} > 0$. There exists a natural involution, known as the Cartan involution, defined by

$$\tau : (\vec{H}, E_{\vec{\alpha}}) \rightarrow -(\vec{H}, E_{-\vec{\alpha}}) \,, \tag{2.1}$$

which can extended to the group by defining $\tau(g_1 g_2) = \tau(g_1)\tau(g_2)$. To construct the non-linear realisation we must specify a subgroup H (not to be confused with the generators of the Cartan subgroup which are denoted by $\vec{H}$). For us this is defined to be the subgroup left invariant under the Cartan involution, i.e. $H = \{g \in G : \tau(g) = g\} \equiv I(G)$. In terms of the Lie algebra $Lie(I(G))$ it is all elements A such that $A = \tau(A)$.

The non-linear realisation is constructed from group elements $g(x) \in G$ which in physical applications depend on the spacetime coordinates x^μ. These are subject to the transformations

$$g(x) \rightarrow g_0 g(x) h^{-1}(x) \tag{2.2}$$

where $h(x) \in H$ and also depends on spacetime. We may write the group element in the form $g(x) = e^{\sum_{\vec{\alpha}>0} \chi_{\vec{\alpha}} E_{\vec{\alpha}}} e^{-\frac{1}{\sqrt{2}} \vec{\phi} \cdot \vec{H}} e^{\sum_{\vec{\alpha}>0} u_{\vec{\alpha}} E_{-\vec{\alpha}}}$, but using the local transformation we can bring it to the form

$$g(\xi) = e^{\sum_{\vec{\alpha}>0} \chi_{\vec{\alpha}} E_{\vec{\alpha}}} e^{-\frac{1}{\sqrt{2}} \vec{\phi} \cdot \vec{H}} \,. \tag{2.3}$$

Here we use $\xi = (\vec{\phi}, \chi_{\vec{\alpha}})$ as a generic symbol for all the scalar fields, which are functions of spacetime, that parameterize the coset representative. Under a rigid $g_0 \in G$ transformation $g(\xi) \rightarrow g_0 g(\xi)$ this form for the coset representative is not preserved. However one can make a compensating transformation $h(g_0, \xi) \in H$ that returns $g_0 g(\xi)$ into the form of equation (2.3);

$$g_0 g(\xi) h^{-1}(g_0, \xi) = g(g_0 \cdot \xi) \,. \tag{2.4}$$

This induces a non-linear action of the group G on the scalars; $\xi \rightarrow g_0 \cdot \xi$.

Using the Cartan involution we can define the notion of a 'generalized transpose';

$$A^{\#} = -\tau(A) \,. \tag{2.5}$$

which in terms of group elements is given by $g^{\#} = (\tau(g))^{-1}$. We note that for group elements $h \in H$ $h^{\#} = h^{-1}$. Unlike τ, the operation $\#$ inverts the order of two group elements i.e. $(g_1 g_2)^{\#} = g_2^{\#} g_1^{\#}$ and also on the product of two elements of the algebra.

Let us now review the construction automorphic forms given in [31]. We will need a linear representation of G. Let $\vec{\mu}^i$, $i = 1, ..., N$ be the weights of the representation and

$|\vec{\mu}^i>$ be a corresponding states. The weights of the representation can be ordered by saying $\vec{\mu}^i > \vec{\mu}^j$ iff $\vec{\mu}^i - \vec{\mu}^j = \vec{\alpha}_{ij}$ is a positive element of the root lattice (note that this requires that one chooses an ordering of the roots). If there are non-zero multiplicities then one can choose any ordering of the degenerate weights that one likes. We will choose to order the weights such that $\vec{\mu}^1 > \vec{\mu}^2 ... > \vec{\mu}^N$. Thus $\vec{\mu}^1$ is the highest weight. The corresponding state satisfies $E_{\vec{\alpha}}|\vec{\mu}^1>= 0$ for all simple roots $\vec{\alpha}$ the states in the rest of the representation are polynomials of $F_{\vec{\alpha}} = E_{-\vec{\alpha}}$ acting on the highest weight state.

We consider states of the form $|\psi> = \sum_i \psi_i |\vec{\mu}^i>$. Under the action $U(g_0)$ of the group G we have

$$|\psi> \rightarrow U(g_0)|\psi> = L(g_0^{-1})\sum_i \psi_i |\vec{\mu}^i> \equiv (U(g_0)\psi_i)|\vec{\mu}^i> = \sum_{i,j} D_i{}^j(g_0^{-1})\psi_j |\vec{\mu}^i> \quad (2.6)$$

where $L(g_0)$ is the expression of the group element g_0 in terms of the Lie algebra elements which now act of the states of the representation in the usual way. We note that the action of the group on the components ψ_i is given by $\psi_i \rightarrow U(g_0)\psi_i = \sum_j D_i{}^j(g_0^{-1})\psi_j$ which is the result expected for a passive action. From this equation we can use the action of the Lie algebra elements on the states of the representation to compute the matrix $D_i{}^j$ of the representation.

Given any linear representation ψ, we can construct the Cartan involution twisted representation, denoted by ψ_τ, which by definition transforms as

$$|\psi_\tau> \rightarrow U(g_0)|\psi_\tau> = L(g_0^{\#})|\psi_\tau> , \quad (2.7)$$

We will also need the dual representation, denoted by $<\psi_D|$, which transforms as

$$<\psi_D| \rightarrow U(g_0)(<\psi_D|) =<\psi_D|L(g_0) , \quad (2.8)$$

It is constructed just so that $<\psi_D|\psi>$ is invariant. Using both constructions we have the dual twisted representation $<\psi_{D\tau}|$ which transforms as $<\psi_{D\tau}| \rightarrow U(g_0) <\psi_{D\tau}| =< \psi_{D\tau}|L(\tau(g_0))$. The representation $<\psi_{D\tau}|$ has the same highest weight as $|\psi>$ and so we can identify it as the same representation [31].

Given any linear realisation, such as the one in equation (2.6), we can construct a non-linear realisation by

$$\begin{aligned}
|\varphi(\xi)> &= \sum \varphi_i(\xi)|\vec{\mu}^i> \\
&= L((g(\xi))^{-1})|\psi> \\
&= e^{\sum_{\vec{\alpha}>0} e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{H}}} e^{-\sum_{\vec{\alpha}>0} \chi_{\vec{\alpha}} E_{\vec{\alpha}}}|\psi>
\end{aligned} \quad , \quad (2.9)$$

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where $g(\xi)$ is the group element of the non-linear realisation in equation (2.3). Under a group transformation $U(g_0)$ it transforms as

$$
\begin{aligned}
U(g_0)|\varphi(\xi)> &= L((g(\xi))^{-1})U(g_0)|\psi> \\
&= L((g(\xi))^{-1})L(g_0^{-1})|\psi> \\
&= L((g_0 g(\xi))^{-1})|\psi> \\
&= L(h^{-1})|\varphi(g_0 \cdot \xi)> ,
\end{aligned}
\tag{2.10}
$$

using equation (2.4). In terms of the component fields we find that $\varphi_i(\xi) = \sum_j D_i{}^j((g(\xi))^{-1})\psi_j$ and $U(g_0)\varphi_i(\xi) = \sum_j D_i{}^j((h)^{-1})\varphi_j(g_0 \cdot \xi)$. The reader can find the examples for $SL(2)$ and $SL(3)$ worked out in equations (5.9), (5.15) and (5.22) respectively.

For the dual and Cartan twisted representations of equations (2.9) and (2.10) we can also introduce non-linearly transforming representations. Indeed for the case of the dual twisted representation we find

$$
< \varphi_{D\tau}(\xi)| =< \psi_{D\tau}|L(\tau(g(\xi))) ,
\tag{2.11}
$$

which transforms as $U(g_0) < \varphi_{D\tau}(\xi)| =< \varphi_{D\tau}(g_0 \cdot \xi)|L(h)$. We note that

$$
< \varphi_{D\tau}(\xi)|\varphi(\xi) > \rightarrow U(g_0)(< \varphi_{D\tau}(\xi)|\varphi(\xi) >) = (< \varphi_{D\tau}(g_0 \cdot \xi)|\varphi(g_0 \cdot \xi) >) ,
\tag{2.12}
$$

An automorphic form $\Phi(\xi)$ is a function on G/H that satisfies

$$
\Phi(g_0 \cdot \xi) = D(h(g_0, \xi)^{-1})\Phi(\xi) ,
\tag{2.13}
$$

for some representation D of H where now the group G is now taken to be a discrete subgroup of G. In this paper we will take the trivial representation and hence have

$$
\Phi(g_0 \cdot \xi) = \Phi(\xi) .
\tag{2.14}
$$

Given a linear representation with highest weight $|\vec{\mu}^1>$ we consider states of the form

$$
|\psi> = \sum_i m_i |\vec{\mu}^i>, \quad m_i \in \mathbf{Z} .
\tag{2.15}
$$

Thus our components are now the integers m_i. We can think of these states as belonging to a lattice

$$
\Lambda = \{\sum_i m_i |\vec{\mu}^i> \mid m_i \in \mathbf{Z} \text{ not all vanishing}\},
\tag{2.16}
$$

with the origin deleted. The discrete version of G that we obtain then consists of elements of G which preserve the lattice. The precise details of this can be rather subtle and we will not comment more on it here.

We then construct the non-linear realisations $\varphi(\xi) >$ using the states of equation (2.13) in equations (2.9) and similarly and $\varphi_{D\tau}(\xi) >$ using equation (2.11); $\phi_{D\tau} = \sum_j <\mu^j|m_j$. The invariant automorphic form is given by

$$\Phi(\xi) = \sum_\Lambda F(u(\xi)) \ . \tag{2.17}$$

where the sum is over the integers m_i that occur in the lattice Λ, F is a function of u and

$$
\begin{aligned}
u(\xi) &=< \varphi_{D\tau}(\xi)|\varphi(\xi) > \\
&=< \psi_{D\tau}|L(\tau(g(\xi)))L((g(\xi))^{-1})|\psi > \\
&= \sum_{i,j} < \vec{\mu}^j|m_j L(\tau(g(\xi)))L((g(\xi))^{-1})m_i|\vec{\mu}^i > \\
&= \sum_{i,j} < \vec{\mu}^j|m_j(e^{-\sum_{\vec{\alpha}>0} E_{-\alpha}\chi_\alpha} e^{\sqrt{2}\vec{\phi}\cdot\vec{H}} e^{-\sum_{\vec{\alpha}>0} E_\alpha\chi_\alpha})m_i|\vec{\mu}^i > \ .
\end{aligned}
\tag{2.18}
$$

Under a group G transformation

$$u(\xi) \to U(g_0)u(\xi) = u(g_0 \cdot \xi') \ . \tag{2.19}$$

However, in the automorphic form of equation (2.17) a $U(g_0)$ transformation on $|\psi >$ is just a rearrangement of the integers m_i and this can be undone by a change of summation of over the lattice. As a result we find that the automorphic form of equation (2.17) transforms as in equation (2.14) as required. The particular case that $F(u) = u^{-s}$ the corresponding automorphic form will simply be denoted by Φ_s. We note that there are additional possible definitions of Eisenstein-like series that we will not discuss here, for example one can construct automorphic forms where where each fundamental weight has a complex number s associated to it. Such forms were recently considered in a String Theory context in [36].

We can construct another automorphic form using $|\psi_\tau >$ and $< \psi_D|$. Let us define

$$v(\xi) =< \varphi_D(\xi)|\varphi_\tau(\xi) >=< \psi_D|L((g(\xi)))L((g(\xi))^\#)|\psi_\tau > \ . \tag{2.20}$$

Using similar arguments once can show that $v(\xi) \to U(g_0)v(\xi) = v(g_0 \cdot \xi')$ and that

$$\Phi^{\tau(R)}(\xi) = \sum_\Lambda F(v) \ . \tag{2.21}$$

is an automorphic form. We will next show that it is related to that of equation (2.17). If we take $F(v) = v^{-s}$ we denote the automorphic form by $\Phi_s^{\tau(R)}(\xi)$.

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We began with the automorphic form constructed from the representation R and write it as

$$\frac{\Gamma(s)}{\pi^s}\Phi_s = \sum_{m_i} \int \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t} m_i A^{ij} m_j} \ , \tag{2.22}$$

where $A = (D((g(\tau(g))^{-1})^{-1})^{ij} = D(\tau(g)g^{-1})^{ij}$. Poisson resuming, using equation (A.2), we find that

$$\frac{\Gamma(s)}{\pi^s}\Phi_s = \sum_{\hat{m}^i} (detA)^{-\frac{1}{2}} \int \frac{dt}{t^{1+(s-N/2)}} e^{-\pi t \hat{m}^i (A^{-1})_{ij} \hat{m}^j} \tag{2.23}$$

We note that $A^{-1} = D(g\tau(g^{-1}))_{ij}$. Changing integration variable t to $1/t$ gives

$$\begin{aligned}
\frac{\Gamma(s)}{\pi^s}\Phi_s &= \frac{\Gamma(\frac{N}{2}-s)}{\pi^{\frac{N}{2}-s}} \sum_{\hat{m}^i} (detA)^{-\frac{1}{2}} \int \frac{dt}{t^{1-s+N/2}} e^{-\frac{\pi}{t}\hat{m}^i (A^{-1}))_{ij} \hat{m}^j} \\
&= \frac{\Gamma(\frac{N}{2}-s)}{\pi^{\frac{N}{2}-s}} (detA)^{-\frac{1}{2}} \sum_{\hat{m}^i} \frac{1}{(\hat{m}^i (A^{-1}))_{ij} \hat{m}^j)^{\frac{N}{2}-s}}
\end{aligned} \tag{2.24}$$

We can interpret this as the automorphic form constructed from the representation $\tau(R)$ as $g \to \tau(g)$ takes $A \to A^{-1}$. We note that if the representation R has highest weight $\vec{\mu}^1$ and lowest weight $\vec{\mu}^N$ then the Cartan involution twisted representation $\tau(R)$ has highest weight $-\vec{\mu}^N$ and lowest weight $-\vec{\mu}^1$. In fact these representations are related by $-W_0$, where W_0 is the unique Weyl reflection with the longest length (see appendix B of reference [31]). The latter is an automorphism of the Dynkin diagram and hence also the Lie algebra and therefore these two representations are related by this automorphism. To give an example, if the representation R of $SL(N)$ is the N representation then the representation $\tau(R)$ is the $\bar{N}$ representation. The automorphism that relates these representations is the one that exchanges the nodes $i \to N - i$ of the $SL(N)$ Dynkin diagram. In this case swapping the fundamental representation associated with node one to that associated with node $N-1$.

In general the sum over the integers $\hat{m}^i$ can be interpreted as the lattice associated with the representation $\tau(R)$ of the group G. As such we have demonstrated that

$$\frac{\Gamma(s)}{\det(D(\tau(g(\xi)))\pi^s}\Phi_s^R = \frac{\Gamma(N/2-s)}{\det(D(g(\xi))\pi^{N/2-s}}\Phi_{N/2-s}^{\tau(R)} \ . \tag{2.25}$$

The automorphic form that appears on the left hand side of this relation is just that of equation (2.17), with $F(u) = \frac{1}{u^s}$ and it is built using the representation R, although it was denoted by just Φ_s there, we above have denoted it by Φ_s^R. The automorphic forms on the right hand side is of the form of that in equation (2.21) and it is built using the representation $\tau(R)$ and so we denoted it by $\Phi_s^{\tau(R)}$ if $F(v) = \frac{1}{v^s}$.

The alert reader will have noticed that we have added and then subtracted the divergent terms at $m_i = 0$ and at $\hat{m}^i = 0$ respectively when carrying out the Poisson

resummation. As such we have assumed that these terms can be regulated and they do not change the result.

3. Evaluation of Automorphic Forms

In this section which wish to develop explicit expressions for the Einstein-like series of Automorphic forms defined in the previous section by

$$\Phi(\xi) = \sum_{\Lambda} \frac{1}{(u(\xi))^s} \ . \tag{3.1}$$

where u is given in equation (2.18). This sum is convergent and Φ is well-defined whenever $s > N/2$ as can be seen heuristically by taking the sum to be an integral. However, it can be defined by analytic continuation for almost all other values of s. As explained in section two we write the states of the representation R as $|\psi> = \sum_i m_i|\vec{\mu}^i>$ so that the state of equation (2.9)

$$|\varphi> = \sum_i m_i e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{H}} e^{-\sum_{\vec{\alpha}} \chi_{\vec{\alpha}} E_{\vec{\alpha}}} |\vec{\mu}^i> \ . \tag{3.2}$$

Since we know the action of the Lie algebra generators on the weights we can evaluate $L(e^{-\sum_{\vec{\alpha}} \chi_{\vec{\alpha}} E_{\vec{\alpha}}})$ on the representation in a straight forward way once we take

$$L(E_{\vec{\alpha}})|\vec{\mu}^i> = c_{\vec{\alpha}i}|\vec{\mu}^i + \vec{\alpha}> \ , \tag{3.3}$$

for some constant $c_{\vec{\alpha}i}$ whose values we will comment later. As a result we find that we can write

$$L(e^{-\sum_{\vec{\alpha}} \chi_{\vec{\alpha}} E_{\vec{\alpha}}})|\vec{\mu}^i> = |\vec{\mu}^i> - \sum_j \tilde{\chi}_{ji}|\vec{\mu}^j> \ . \tag{3.4}$$

which defines the symbols $\tilde{\chi}_{ji}$. Taking the innerproduct with $\omega_k^{-1}|\vec{\mu}^k>$ we find

$$\begin{aligned}
\tilde{\chi}_{ki} &= \delta_{ki} - \omega_k^{-1} <\vec{\mu}^k|L(e^{-\sum_{\vec{\alpha}} \chi_{\vec{\alpha}} E_{\vec{\alpha}}})|\vec{\mu}^i> \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!\omega_k} \sum_{\vec{\alpha}_1}\cdots\sum_{\vec{\alpha}_n} \chi_{\vec{\alpha}_1}\cdots\chi_{\vec{\alpha}_n} <\vec{\mu}^k|L(E_{\vec{\alpha}_1}\ldots E_{\vec{\alpha}_n})|\vec{\mu}^i> \\
&= c_{\vec{\alpha}_{ki}i}\chi_{\vec{\alpha}_{ki}} + Poly(\chi_{\vec{\beta}}, 0 < \vec{\beta} < \vec{\alpha}_{ki}) \ ,
\end{aligned} \tag{3.5}$$

where $\vec{\alpha}_{ki} = \vec{\mu}^k - \vec{\mu}^i$ and $Poly$ is a polynomial in $\chi_{\vec{\beta}}$ that only involves roots such that $0 < \vec{\beta} < \vec{\alpha}_{ki}$. Note also that $\tilde{\chi}_{ki} = 0$ unless $\vec{\alpha}_{ki} = \vec{\mu}^k - \vec{\mu}^i$ is a positive element of the root lattice, *i.e. if* $k < i$. We have normalised the necessarily orthogonal states of the representation as $<\vec{\mu}^k|\vec{\mu}^i> = \omega_i\delta_{ik}$

Thus the state of equation (3.2) is then given by

$$|\varphi> = \sum_i m_i L(e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{H}}) \left(|\vec{\mu}^i> - \sum_k \tilde{\chi}_{ki}|\vec{\mu}^k> \right)$$
$$= \sum_i m_i \left(e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i}|\vec{\mu}^i> - \sum_k \tilde{\chi}_{ki}e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^k}|\vec{\mu}^k> \right) \quad . \tag{3.6}$$

Rearranging the sum over i this can be rewritten as

$$|\varphi> = \sum_i \left(m_i - \sum_{j>i} \tilde{\chi}_{ij}m_j \right) e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i}|\vec{\mu}^i> \quad . \tag{3.7}$$

Thus we find the u of the automorphic form of equation (2.18) is given

$$u = \sum_{i=1}^N (m_i - \tilde{\chi}_i)^2 \omega_i e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i} \quad , \tag{3.8}$$

where

$$\tilde{\chi}_i = \sum_{j>i} \tilde{\chi}_{ij}m^j \quad . \tag{3.9}$$

Note that all the dependence of the axion-like fields $\chi_{\vec{\alpha}}$ are contained in the $\tilde{\chi}_i$. Since $\tilde{\chi}_{ij} = 0$ if $i > j$ we find that $\tilde{\chi}_j$ depends only on m_k, $k > j$. In particular the only dependence on m_1 in u is the explicit m_1 in the first term.

In the above we have introduced a normalisation of the states ω_i and a constant $c_{\vec{\alpha} i}$ in the action of the E_α generators of equation (3.3). Clearly, we are free to choose one or the other of these constants by scaling the states of the representation. In particular if we choose $\omega_i = 1$ then we can set $L(E_{\vec{\alpha}_a})|\vec{\mu}^i> = N_{\alpha_a}|\vec{\mu}^i + \vec{\alpha}_a>$ for any simple root α_a and then determine all the constants N_{α_a} by implementing the Lie algebra relations. For example if one does this for $SL(N)$ one finds that all the $N_{\alpha_a} = 0, 1$ but this will not be the case in general.

It will prove useful to write, using equation (A.1), the automorphic form as

$$\Phi = \sum_\Lambda \frac{1}{u^s} = \sum_\Lambda \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t}u} \quad , \tag{3.10}$$

We can split the sum into two pieces; a first terms that is over only $m_1 \neq 0$ with all other $m_j = 0$ for $j > 1$ and a second term that is over over all m_1, including zero, and over all m_j, $j > 1$, but not including $m_j = 0$, $j = 2, 3, \ldots$. The result is

$$\Phi_s^N = \sum_{m_1 \neq 0} \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t}m_1^2 \omega_1 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1}}$$
$$+ \sum_{m_1=-\infty}^\infty \sum_{\Lambda_1} \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t}(m_1-\tilde{\chi}_1)^2 \omega_1 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1}} e^{-\frac{\pi}{t}u_1} \quad , \tag{3.12}$$

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where

$$u_1 = \sum_{i>1} (m_i - \tilde{\chi}_i)^2 \omega_i e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i} \ . \tag{3.13}$$

and Λ_1 is the lattice spanned by $m_2, ..., m_N \in \mathbf{Z}$, with the origin omitted. We note that there is no $\tilde{\chi}_1$ in the first term since it depends on m_j, $j > 1$, but in this term we have $m_j = 0$ for $j > 1$. We have dented Φ by Φ_s^N for reasons that will become apparent; the N refers to the dimension of the lattice.

The first term is simply evaluated using equation (A.1). While the sum in the second term is over all integers m_1 and so we may use Poisson resummation, that is equation (A.2). Carrying this out we find that

$$\Phi_s^N = 2\zeta(2s)\frac{1}{\omega_1^s}e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\mu}^1}$$
$$+ \sum_{\hat{m}_1=-\infty}^{\infty}\sum_{\Lambda_1}\frac{\pi^s}{\Gamma(s)}\int_0^{\infty}\frac{dt}{t^{1+s}}\sqrt{\frac{t}{\omega_1}}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1}e^{2\pi i\hat{m}_1\tilde{\chi}_1} \tag{3.14}$$
$$\times e^{-\pi t\hat{m}_1^2\omega_1^{-1}e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1}}e^{-\frac{\pi}{t}u_1} \ .$$

We will now construct a recursion relation in N and s by splitting the second term into a piece corresponding to $\hat{m}_1 = 0$ and the rest

$$\Phi_s^N = 2\zeta(2s)\frac{1}{\omega_1^s}e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\mu}^1}$$
$$+ \sum_{\Lambda_1}\sqrt{\frac{\pi}{\omega_1}}\frac{\Gamma(s-1/2)}{\Gamma(s)}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1}\frac{\pi^{s-1/2}}{\Gamma(s-1/2)}\int_0^{\infty}\frac{dt}{t^{1+(s-1/2)}}e^{-\frac{\pi}{t}u_1}$$
$$+ \frac{2}{\sqrt{\omega_1}}\sum_{\hat{m}_1=1}^{\infty}\sum_{\Lambda_1}\cos(2\pi\hat{m}_1\tilde{\chi}_1)\frac{\pi^s}{\Gamma(s)}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1}\int_0^{\infty}\frac{dt}{t^{1+(s-1/2)}}e^{-\pi t\hat{m}_1^2\omega_1^{-1}e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1}}e^{-\frac{\pi}{t}u_1}$$
$$= 2\zeta(2s)\frac{1}{\omega_1^s}e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\mu}^1} + \sqrt{\frac{\pi}{\omega_1}}\frac{\Gamma(s-1/2)}{\Gamma(s)}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1}\Phi_{s-1/2}^{N-1} + \Upsilon_s^N \ . \tag{3.15}$$

Here $\Phi_{s-1/2}^{N-1}$ takes the form of the original function Φ_s^N but with a shifted valued of s and a lattice Λ_1 which is constructed only from the basis of $N-1$ states $|\vec{\mu}^2>, ..., |\vec{\mu}^N>$. Using the Bessel function integral formula (A.3) we can write Υ_s^N as

$$\Upsilon_s^N = \frac{4}{(\sqrt{\omega_1})^{s+1/2}}\frac{\pi^s}{\Gamma(s)}e^{-\frac{1}{\sqrt{2}}(s+1/2)\vec{\phi}\cdot\vec{\mu}^1}\sum_{\hat{m}_1=1}^{\infty}\sum_{\Lambda_1}\left(\frac{\hat{m}_1}{\sqrt{u_1}}\right)^{s-1/2}\cos(2\pi\hat{m}_1\tilde{\chi}_1)$$
$$\times K_{s-1/2}\left(\frac{2\pi}{\sqrt{\omega_1}}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1}\hat{m}_1\sqrt{u_1}\right) \ . \tag{3.17}$$

In the limit of large x, $K_{s-1/2}(x) \sim e^{-x}$ is exponentially small. Thus Υ_s^N is exponentially suppressed in some region of moduli space and we refer to it as non-perturbative.

Substituting into equation (3.15) we now find the recursion relation for the remaining, perturbative, piece

$$\Phi^N_{s\ p} = 2\zeta(2s)\frac{1}{\omega^s_1}e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\mu}^1} + \sqrt{\frac{\pi}{\omega_1}}\frac{\Gamma(s-1/2)}{\Gamma(s)}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1}\Phi^{N-1}_{s-1/2p}. \qquad (3.18)$$

We can iterate this recursion relation N-times to obtain

$$\Phi_p = \sum_{k=1}^{N}\frac{2}{\omega_k^{s-\frac{k-1}{2}}}\zeta(2s-k+1)\pi^{\frac{k-1}{2}}\frac{\Gamma(s-\frac{k-1}{2})}{\Gamma(s)}e^{-\sqrt{2}(s-\frac{k-1}{2})\vec{\phi}\cdot\vec{\mu}^k}\prod_{i<k}\sqrt{\frac{1}{\omega_i}}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i}\ . \quad (3.19)$$

We also obtain a recursion relation for the remaining non-perturbative part of Φ by including the Υ-terms:

$$\Phi^N_{s\ np} = \sqrt{\frac{\pi}{\omega_1}}\frac{\Gamma(s-1/2)}{\Gamma(s)}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^1}\Upsilon^{N-1}_{s-1/2} + \Upsilon^N_s\ . \qquad (3.20)$$

Iterating this gives

$$\begin{aligned}\Phi_{np} = \sum_{k=1}^{N-1}&\frac{4}{(\sqrt{\omega_k})^{s-\frac{k-2}{2}}}\frac{\pi^s}{\Gamma(s)}e^{-\frac{1}{\sqrt{2}}(s-\frac{k-2}{2})\vec{\phi}\cdot\vec{\mu}^k}\prod_{j<k}\sqrt{\frac{1}{\omega_j}}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^j}\cos(2\pi\hat{m}_k\tilde{\chi}_k)\\ \times&\sum_{\hat{m}_k=1}^{\infty}\sum_{\Lambda_k}\left(\frac{\hat{m}_k}{\sqrt{u_k}}\right)^{s-\frac{k}{2}}K_{s-\frac{k}{2}}\left(\frac{2\pi}{\sqrt{\omega_k}}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^k}\hat{m}_k\sqrt{u_k}\right)\end{aligned}\ , \quad (3.21)$$

where Λ_k is spanned by $|\vec{\mu}^k>,...,|\vec{\mu}^N>$ and

$$u_k = \sum_{i>k}(m_i - \tilde{\chi}_i)^2\omega_i e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i}\ . \qquad (3.22)$$

We see that all the dependence on $\chi_{\vec{\alpha}}$ is contained in the Φ_{np}. We have therefore split Φ into a perturbative and a non-perturbative piece, $\Phi = \Phi_p + \Phi_{np}$. We will see later when comparing with string theory that this is indeed the correct split.

Equation (3.19) is the main technical result of this paper. It shows that the automorphic forms constructed above through equation (3.1) for any N-dimensional representation of G always have N contributions to their perturbative part. However, as we will see below, this does not in general correspond to N distinct orders of string perturbation theory. These N terms are given by exponentials of the form $e^{\frac{1}{\sqrt{2}}\vec{w}\cdot\vec{\phi}}$ where $\vec{w}$ are certain linear combinations of the weights of G. Thus determining the perturbative part is essentially reduced to a linear algebra problem in the weight space of the representation that is used to construct the automorphic form.

4. Regularization of Automorphic forms

As we mentioned above Φ_s is well defined when $s > N/2$. However the most well studied higher derivative corrections correspond to small values of s such as $s = 3/2$ for the R^4 term. Thus we wish to extend the definition of Φ_s to more general values of s. One can show that Φ_{np} is always convergent and hence well defined. The divergences show up in Φ_p because $\zeta(z)$ and $\Gamma(z)$ are only convergent for $z > 1$ and $z > 0$ respectively. However since these functions can be analytically continued we will be able regulate Φ and extend its definition to any value of s. Regularization has also been discussed for some special cases in [27].

More specifically by analytic continuation $\zeta(z)$ can be defined for all $z \neq 1$ and $\Gamma(z)$ for all $z \neq 0, -1, -2, \dots$. These are simple poles with

$$\zeta(1 + \epsilon) = \frac{1}{\epsilon} \qquad \Gamma(-n + \epsilon) = \frac{(-1)^n}{n!\epsilon} \ , \tag{4.1}$$

and a useful formula is

$$\zeta(-z) = -2^{-z}\pi^{-z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1 + z)\zeta(1 + z) \ . \tag{4.2}$$

Note that $\zeta(-2n) = 0$ and hence $\zeta(-2n)\Gamma(-n)$ is finite for $n = 1, 2, 3\dots$ Thus for generic values of s the automorphic forms are well-defined using analytic continuation. However for some values, namely when $s \leq N/2$ with $2s$ a positive integer, we encounter divergences. To regulate the automorphic form we can deform $s \to s + \epsilon$, remove the $1/\epsilon$ pole and then take the limit $\epsilon \to 0$. This will preserve the automorphic properties of Φ, provided that the residue of the $1/\epsilon$ pole is a constant.

First consider the case $s < N/2$ with $2s$ a positive integer. The terms in Φ_p come with coefficients

$$\zeta(2s - k + 1)\Gamma(s - (k - 1)/2)/\Gamma(s) \ .$$

There are potential divergences when $2s - k + 1 = 1$ or $s - (k - 1)/2 = 0, -1, -2, -3, \dots$. However since $\zeta(-2n)\Gamma(-n)$ is finite if $n = 1, 2, 3, \dots$ we see that the only problematic terms in the later case arise when $s - (k - 1)/2 = 0$. Thus there are two divergent terms, one where $k = 2s$ and one with $k = 2s + 1$. Let us deform $s \to s + \epsilon$ and look at these two terms. We find

$$2\zeta(1 + 2\epsilon)\frac{\Gamma(1/2 + \epsilon)}{\Gamma(s + \epsilon)}\pi^{s-1/2}e^{-\sqrt{2}(1/2+\epsilon)\vec{\phi}\cdot\vec{\mu}^{2s}} \prod_{i<2s} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i}$$

$$+ 2\zeta(2\epsilon)\frac{\Gamma(\epsilon)}{\Gamma(s + \epsilon)}\pi^s e^{-\sqrt{2}\epsilon\vec{\phi}\cdot\vec{\mu}^{2s+1}} \prod_{i<2s+1} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i} \ . \tag{4.3}$$

Substituting in $\zeta(0) = -1/2$, $\Gamma(1/2) = \sqrt{\pi}$ and extracting the $1/\epsilon$ poles we find

$$\left(\frac{1}{\epsilon}\frac{\Gamma(1/2)}{\Gamma(s)}\pi^{s-1/2} + 2\frac{1}{\epsilon}\frac{\zeta(0)}{\Gamma(s)}\pi^s\right) \prod_{i<2s+1} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i} + \mathcal{O}(\epsilon^0) \tag{4.4}$$

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which vanishes since $\zeta(0) = -1/2$, $\Gamma(1/2) = \sqrt{\pi}$. Thus Φ_p is finite in the limit $\epsilon \to 0$. The effect of this is to remove the two divergent terms from Φ_p and replace them by

$$-\frac{2\pi^s}{\Gamma(s)}\left(\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}^{2s} - \vec{\mu}^{2s+1}) - (\gamma - \ln(4\pi))\right)\prod_{i<2s+1}e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i}\ , \qquad (4.5)$$

which come from the $\mathcal{O}(\epsilon^0)$ part of the regularlisation. Here γ is the Euler constant. Note that the effect of this is that the two terms, which are both proportional to $e^{-\frac{1}{\sqrt{2}}\phi\cdot(\vec{\mu}^1 + ... \vec{\mu}^{2s})}$, change from having a divergent coefficient to a finite one that depends linearly on $\vec{\phi}\cdot(\vec{\mu}^{2s} - \vec{\mu}^{2s+1})$.

When $s = N/2$ there is a $\zeta(1)$ term at $k = N$ but no $\Gamma(0)$ term. Thus we only find the first contribution to the pole. However we see that, since $\sum_{k=1}^N \vec{\mu}^k = \vec{0}$, the residue of the $1/\epsilon$ pole is a constant. In particular we find

$$\frac{1}{\epsilon}\frac{\pi^{N/2}}{\Gamma(N/2)} - \sqrt{2}\frac{\pi^{N/2}}{\Gamma(N/2)}\vec{\phi}\cdot\vec{\mu}^N + \frac{\pi^{N/2}}{\Gamma(N/2)}(\gamma - 2\ln(2) - \Gamma'(N/2)/\Gamma(N/2)) + \mathcal{O}(\epsilon)\ . \quad (4.6)$$

Thus we can obtain a well-defined automorphic form by taking

$$\Phi_{s=N/2} = \lim_{\epsilon \to 0}\left(\Phi_{N/2+\epsilon} - \frac{1}{\epsilon}\frac{\pi^{N/2}}{\Gamma(N/2)} - \frac{\pi^{N/2}}{\Gamma(N/2)}(\gamma - 2\ln(2) - \Gamma'(N/2)/\Gamma(N/2))\right) \quad (4.7)$$

Note that we are also free to remove an arbitrary finite constant from $\Phi_{s+\epsilon}$. Thus in this case the effect of this is to remove the final, divergent, term from Φ_p and replace it with

$$-\sqrt{2}\frac{\pi^{N/2}}{\Gamma(N/2)}\vec{\phi}\cdot\vec{\mu}^N\ . \qquad (4.8)$$

There is another interesting case where $s = -n$ for some $n = 0, 1, 2, ...$ Here the $\Gamma(s)$ term in the denominator diverges so

$$\Phi_{s=-n} = 0\ . \qquad (4.9)$$

However the $k = 1$ term in Φ_p survives when $s = 0$ (recall that $\zeta(2s)\Gamma(s)$ is finite when s is a negative integer) and we therefore find

$$\Phi_{s=0} = -1 \qquad (4.10)$$

for any representation.

5. Elementary Examples

To illustrate the general formalism developed above we now work it out for the simplest cases, namely $SL(2)$ and $SL(3)$.

5.1 $SL(2)$

The simplest example of an automorphic form is for the group $SL(2)$ with generators

$$E_{\beta_1}, H, F_{\beta_1} ,\tag{5.1}$$

with a single positive root generator E_{β_1} and a single negative root generator $F_{\beta_1} = E_{-\beta_1}$ where $\beta_1 = \sqrt{2}$. The fundamental weight dual to β_1 is just $\mu^1 = 1/\sqrt{2}$. We take the local subgroup to be the Cartan involution invariant subgroup which is generated by $E_{\beta_1} - F_{\beta_1}$. As a result we may choose the coset representative to be given by

$$g = e^{\chi E_{\beta_1}} e^{-\frac{1}{\sqrt{2}}\phi H} ,\tag{5.2}$$

The 2-dimensional representation of $SL(2)$ has highest weight μ^1 and lowest weight $\mu^2 = -\mu^1$. Taking the states to be normalised so that $E|\mu^2> = |\mu^1>$, we find the action of the Lie algebra elements on the representation is given by

$$E_{\beta_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad F_{\beta_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .\tag{5.3}$$

Similarly acting with $L(g)$ on the states of the representation we find that

$$L(g)(m_1|\mu^1> + m_2|\mu^2>) = (e^{-\phi/2}m_1 + m_2 e^{\phi/2}\chi)|\mu^1> + m_2 e^{\phi/2}|\mu^2>$$
$$L(g^{-1})(m_1|\mu^1> + m_2|\mu^2>) = (e^{\phi/2}m_1 - m_2 e^{\phi/2}\chi)|\mu^1> + m_2 e^{-\phi/2}|\mu^2> .\tag{5.4}$$

Thus acting on the m_1 and m_2 as a column vector we find that $L(g)$ and $L(g^{-1})$ are represented by the familiar matrices

$$e^{\phi/2}\begin{pmatrix} e^{-\phi} & \chi \\ 0 & 1 \end{pmatrix} \qquad e^{\phi/2}\begin{pmatrix} 1 & -\chi \\ 0 & e^{-\phi} \end{pmatrix} .$$

respectively. One can explicitly compute the non-linear transformation by taking a constant element

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2) ,\tag{5.5}$$

one then computes $g_0 g$ and finds a matrix

$$h = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2) ,\tag{5.6}$$

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such that $g_0 g(\phi, \chi) h^{-1} = g(\phi', \chi')$ is again of the form (5.4) but with $\phi \to \phi'$ and $\chi \to \chi'$. If we introduce the complex field $\tau = -\chi + ie^{-\phi}$ then we find

$$e^{i\theta} = \frac{c\tau + d}{|c\tau + d|} \; , \tag{5.7}$$

and recover the familiar fractional linear transformation

$$\tau' = \frac{a\tau + b}{c\tau + d} \; . \tag{5.8}$$

As explained in section 2, to construct the automorphic form we first compute the state $|\varphi>$ of equation (2.9):

$$|\varphi>= L(g^{-1})(m_1|\mu^1 > + m_2|\mu^2 >) = (m_1 e^{\phi/2} - m_2 \chi e^{\phi/2})|\mu^1 > + m_2 e^{-\phi/2}|\mu^2 > \; , \tag{5.9}$$

Using equation (2.18) we find the automorphic form to be given by

$$\Phi_s = \sum_{m_1, m_2 \neq (0,0)} \frac{1}{[(m_1 - \chi m_2)^2 e^\phi + m_2^2 e^{-\phi}]^s} \; . \tag{5.10}$$

Using equation (3.19) we find that the perturbative part is given by [19]

$$\Phi_p = 2\zeta(2s)e^{-s\phi} + 2\zeta(2s - 1)\pi^{\frac{1}{2}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} e^{(s-1)\phi} \; . \tag{5.11}$$

We note that in this simple case $u_1 = m_2^2 e^{-\phi}$ and $\tilde{\chi}_1 = \chi m_2$ so that, using the equation (3.21) the non-perturbative part is given by [19]

$$\Phi_{np} = 4 \frac{\pi^s}{\Gamma(s)} e^{-\phi/2} \sum_{\hat{m}_1 = 1}^{\infty} \sum_{m_2 \neq 0} \left(\frac{\hat{m}_1}{|m_2|} \right)^{s - \frac{1}{2}} \cos(2\pi \hat{m}_1 m_2 \chi) K_{s - \frac{1}{2}}(2\pi \hat{m}_1 |m_2| e^{-\phi}) \; . \tag{5.12}$$

We can also consider the 3-dimensional representation of $SL(2)$. The root string in this case is

$$\{\vec{\mu}^1, \vec{\mu}^2, \vec{\mu}^3\} = \{\sqrt{2}, 0, -\sqrt{2}\} \; , \tag{5.13}$$

i.e. the highest weight is $\vec{\mu}^1 = \sqrt{2}$. In particular we find

$$\tilde{\chi}_{12} = \tilde{\chi}_{23} = \chi \qquad \tilde{\chi}_{13} = -\frac{1}{2}\chi^2 \; . \tag{5.14}$$

Calculating the non-linearly realised state of equation (2.9) we find that

$$
\begin{aligned}
|\varphi> &= L(g^{-1})(m_1|\vec{\mu}^1 > + m_2|\vec{\mu}^2 > + m_3|\vec{\mu}^3 >) \\
&= (m_1 - m_2\chi + \frac{1}{2}m_3\chi^2)e^\phi|\vec{\mu}^1 > + (m_2 - m_3\chi)|\vec{\mu}^2 > + m_3 e^{-\phi}|\vec{\mu}^3 >) \; .
\end{aligned}
\tag{5.15}
$$

Then the automorphic form of equation (2.19) becomes

$$\Phi_s = \sum_{m_1,m_2,m_3 \neq (0,0,0)} \frac{1}{[(m_1 - \chi m_2 + \frac{1}{2}\chi^2 m_3)^2 e^{2\phi} + (m_2 - \chi m_3)^2 + m_3^2 e^{-2\phi}]^s} \cdot \qquad (5.16)$$

Using equation (3.19) the perturbative part is given by

$$\Phi_p = 2\zeta(2s)e^{-2s\phi} + 2\zeta(2s-1)\pi^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}e^{-\phi} + 2\pi\frac{\Gamma(s-1)}{\Gamma(s)}\zeta(2s-2)e^{(2s-3)\phi} \cdot \qquad (5.17)$$

and equation (3.21) the non-perturbative piece by

$$\Phi_{np} = 4\frac{\pi^s}{\Gamma(s)}e^{-(s+1/2)\phi}\sum_{\hat{m}_1=1}^{\infty}\sum_{(m_2,m_3)\neq(0,0)}\left(\frac{\hat{m}_1}{\sqrt{u_1}}\right)^{s-\frac{1}{2}}\cos(2\pi\hat{m}_1\tilde{\chi}_1)K_{s-\frac{1}{2}}(2\pi\hat{m}_1 e^{-\phi}\sqrt{u_1})$$
$$+ 4\frac{\pi^s}{\Gamma(s)}e^{-\phi}\sum_{\hat{m}_2=1}^{\infty}\sum_{m_3\neq 0}\left(\frac{\hat{m}_2}{\sqrt{u_2}}\right)^{s-1}\cos(2\pi\hat{m}_2\tilde{\chi}_2)K_{s-1}(2\pi\hat{m}_2\sqrt{u_2})$$

$$\qquad (5.18)$$

Here we note that $\tilde{\chi}_1 = \chi m_2 + \chi^2 m_3$, $\tilde{\chi}_2 = \chi m_3$, $u_1 = (m_2 - \chi m_3)^2 + m_3^2 e^{-2\phi}$ and $u_2 = m_3^2 e^{-2\phi}$. Thus these terms are exponentially suppressed as $e^{\phi} \to \infty$.

5.2 $SL(3)$

Let us now consider the 3-dimensional representation of $SL(3)$. The simple roots are

$$\vec{\alpha}_1 = (\sqrt{2}, 0) \qquad \vec{\alpha}_2 = (-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}) \,, \qquad (5.19)$$

and hence the fundamental weights are

$$\vec{\lambda}^1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}) \qquad \vec{\lambda}^2 = (0, \sqrt{\frac{2}{3}}) \,. \qquad (5.20)$$

The 3-dimensional representation has highest weight $\vec{\mu}^1 = \vec{\lambda}^1$ and the root string is

$$\vec{\mu}^1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}) \qquad \vec{\mu}^2 = \vec{\mu}^1 - \vec{\alpha}_1 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}) \qquad \vec{\mu}^3 = \vec{\mu}^1 - \vec{\alpha}_1 - \vec{\alpha}_2 = (0, -\sqrt{\frac{2}{3}}) \,. \qquad (5.21)$$

The fields ξ consist of $\vec{\phi} = (\phi, \rho)$ associated to the Cartan subalgebra and $\chi_{\vec{\alpha}_1}$, $\chi_{\vec{\alpha}_1}$ and $\chi_{\vec{\alpha}_1+\vec{\alpha}_2}$ are the three axion-like modes associated to the raising operators $E_{\vec{\alpha}_1}, E_{\vec{\alpha}_2}$ and $E_{\vec{\alpha}_1+\vec{\alpha}_2}$. We now find

$$|\varphi> = L(g^{-1})(m_1|\vec{\mu}^1> + m_2|\vec{\mu}^2> + m_3|\vec{\mu}^3>)$$
$$= (m_1 - m_2\chi_{\vec{\alpha}_1} - m_3(\chi_{\vec{\alpha}_1+\vec{\alpha}_2} - \frac{1}{2}\chi_{\vec{\alpha}_1}\chi_{\vec{\alpha}_2}))e^{\phi/2+\rho/2\sqrt{3}}|\vec{\mu}^1> \qquad (5.22)$$
$$+ (m_2 - m_3\chi_{\vec{\alpha}_2})e^{-\phi/2+\rho/2\sqrt{3}}|\vec{\mu}^2> + m_3 e^{-\rho/\sqrt{3}}|\vec{\mu}^3>) \,.$$

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Following (2.18) we find that the automorphic form is given by

$$\Phi_s = \sum_{m_1,m_2,m_3 \neq (0,0,0)} \frac{1}{<\varphi|\varphi>^s} . \tag{5.23}$$

Using equation (3.19) the perturbative part is given by

$$\begin{aligned}
\Phi_p &= 2\zeta(2s)e^{-\sqrt{2}s\vec{\phi}\cdot\vec{\mu}^3} + 2\zeta(2s-1)\pi^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}e^{-\sqrt{2}\vec{\phi}\cdot((s-1/2)\vec{\mu}^2+1/2\vec{\mu}^3)} \\
&\quad + 2\pi\frac{\Gamma(s-1)}{\Gamma(s)}\zeta(2s-2)e^{-\sqrt{2}\vec{\phi}\cdot((s-1)\vec{\mu}^1+1/2\vec{\mu}^2+1/2\vec{\mu}^3)} \\
&= 2\zeta(2s)e^{-s\phi}e^{-s\rho/\sqrt{3}} + 2\zeta(2s-1)\pi^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}e^{(s-1)\phi}e^{-s\rho/\sqrt{3}} \\
&\quad + 2\pi\frac{\Gamma(s-1)}{\Gamma(s)}\zeta(2s-2)e^{(2s-3)\rho/\sqrt{3}}
\end{aligned} \tag{5.24}$$

While equation (3.21) gives the non-perturbative part to be

$$\begin{aligned}
\Phi_{np} &= 4\frac{\pi^s}{\Gamma(s)}e^{-\frac{1}{2}(s+1/2)(\phi+\rho/\sqrt{3})}\sum_{\hat{m}_1=1}^{\infty}\sum_{(m_2,m_3)\neq(0,0)}\left(\frac{\hat{m}_1}{\sqrt{u_1}}\right)^{s-\frac{1}{2}} \\
&\quad \times \cos(2\pi\hat{m}_1\tilde{\chi}_1)K_{s-\frac{1}{2}}(2\pi\hat{m}_1\sqrt{u_1}e^{-\frac{1}{2}(\phi+\rho/\sqrt{3})}) \\
&\quad + 4\frac{\pi^s}{\Gamma(s)}e^{\frac{1}{2}(s-1)\phi-\frac{1}{2}(s+1)\rho/\sqrt{3}}\sum_{\hat{m}_2=1}^{\infty}\sum_{m_3\neq0}\left(\frac{\hat{m}_2}{\sqrt{u_2}}\right)^{s-1} \\
&\quad \times \cos(2\pi\hat{m}_2\tilde{\chi}_2)K_{s-1}(2\pi\hat{m}_2\sqrt{u_2}e^{\frac{1}{2}(\phi-\rho/\sqrt{3})})
\end{aligned} \tag{5.25}$$

where

$$u_1 = (m_2 - \chi_{\vec{\alpha}_2}m_3)^2 e^{-\phi+\rho/\sqrt{3}} + m_3^2 e^{-2\rho/\sqrt{3}} , \qquad u_2 = m_3^2 e^{-2\rho/\sqrt{3}} , \tag{5.26}$$

and

$$\tilde{\chi}_1 = \chi_{\vec{\alpha}_1}m_2 + (\chi_{\vec{\alpha}_1+\vec{\alpha}_2} - \frac{1}{2}\chi_{\vec{\alpha}_1}\chi_{\vec{\alpha}_2})m_3 \qquad \tilde{\chi}_2 = \chi_{\vec{\alpha}_2}m_3 . \tag{5.27}$$

6. Perturbative contribution for E_{n+1} Automorphic Forms in terms of $SL(2) \otimes SL(n)$

The main purpose for studying the automorphic forms in this paper is to find suitable examples that could occur in the higher derivative corrections for String Theory compactified to $d = 10 - n$ dimensions on an n-dimensional torus. Examining these automorphic forms in detail should give detailed information on the quantum string corrections. In

section three, in equation (3.18), we found the "perturbative" contribution from the automorphic form for any group G and any representation. The result is expressed in terms of weights of the representation and in order to compare to string theory we must express it in terms of string variables such the string coupling relevant to that dimension and other quantities such as the volume of the compactified space. In this section we give a more explicit form for the perturbative part by taking explicit expressions for the weights involving quantities that can then be related to string quantities. This final step and the comparison to string theory will be given in section seven.

The E_{n+1} symmetry that arises in the compactification of the IIB theory on an n torus has the Dynkin diagram of figure 1
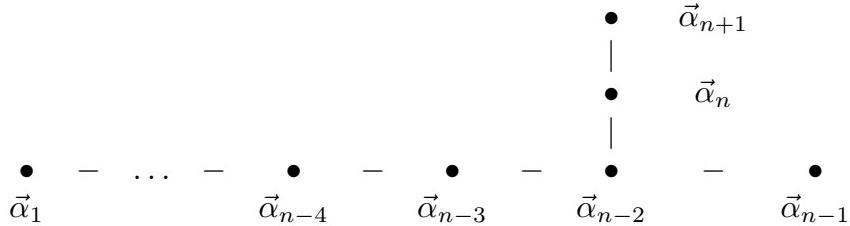

FIGURE 1

The node labeled $n+1$ corresponds to the $SL(2)$ symmetry of the IIB theory while the nodes labeled 1 to $n-1$ arise are associated with the part of the gravity of the IIB theory compactified on the n torus. A natural decomposition of the E_{n+1} symmetry is found by deleting the node labeled n and analysing the representations of E_{n+1} that we need in terms of the resulting $SL(2) \otimes SL(n)$ algebra. This deletion is shown in figure 2.
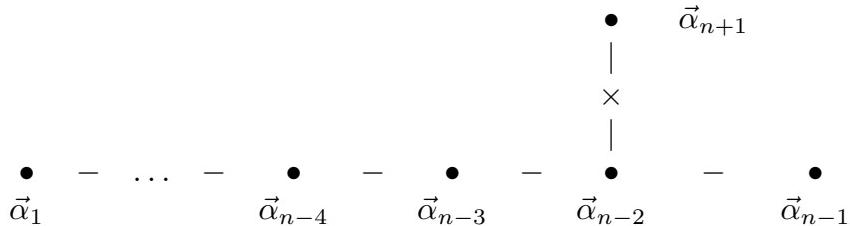

FIGURE 2

This decomposition is very natural when the IIB theory is considered from the perspective of E_{11} [39] and [40,41] contain explicit decompositions of the relevant representations. In addition a review on U-duality discussing E_{n+1} representations is [42]. Deleting node 10 of the E_{11} Dynkin diagram leads to the $SL(10)$ content of the ten dimensional E_{11} theory and at low levels this is just the content of the IIB supergravity theory [43]. The theory in d dimensions arises from the E_{11} theory by deleting node d of the E_{11} Dynkin

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diagram leaving the Dynkin diagrams corresponding to the E_{n+1} symmetry and $SL(d)$ corresponding to the gravity in the remaining spacetime.

The representations of interest to us are the fundamental representation associated with node $n + 1$. In this section we will use the above decomposition to find the explicit form of the weights of this representation in terms of those of $SL(2) \otimes SL(n)$ and $\vec{\phi} = (\phi, \rho, \underline{\phi})$. Here ϕ is associated to the Cartan subalgebra of $SL(2)$, $\underline{\phi}$ to the Cartan subalgebra of $SL(n)$ and ρ is associated with the Cartan generator of E_{n+1} associated with node n and so does not appear in $SL(2) \otimes SL(n)$. In section seven we will relate these fields to the string variables. By substituting these into the formulae we will obtain the perturbative contribution in a form that allows its terms to be compared with string theory in d dimensions.

Although we will not use any E_{11} input we will carry out the decomposition using the techniques employed in the analysis of this algebra [39-44]. As a first step we write the simple roots of E_{n+1} as

$$\vec{\alpha}_{n+1} = (\beta_1, 0, 0), \qquad \vec{\alpha}_n = (0, x, \underline{0}) - \vec{\nu}, \qquad \vec{\alpha}_i = (0, 0, \underline{\alpha}_i), \qquad (6.1)$$

where $\vec{\nu} = (\mu_1, 0, \underline{0}) + (0, 0, \underline{\lambda}^{n-2})$ and $i = 1, \ldots, n-1$. In these equations $\underline{\alpha}_i$ and $\underline{\lambda}^i$ are the simple roots and fundamental weights of $SL(n)$ and $\beta_1 = \sqrt{2}$ and $\mu_1 = \frac{1}{\sqrt{2}}$ the simple root and fundamental weight of $SL(2)$. Demanding that $\vec{\alpha}_n^2 = 2$ we find that

$$x = \sqrt{\frac{8 - n}{2n}} . \qquad (6.2)$$

We will also need the fundamental weights of E_{n+1}, denoted $\vec{\lambda}^a, a = 1, \ldots, n+1$. These are readily determined to be

$$\vec{\lambda}^i = \left(0, \frac{1}{x} \underline{\lambda}^{n-2} \cdot \underline{\lambda}^i, \underline{\lambda}^i\right), \qquad \vec{\lambda}^n = \left(0, \frac{1}{x}, \underline{0}\right), \qquad \vec{\lambda}^{n+1} = \left(\mu_1, \frac{1}{2x}, \underline{0}\right). \qquad (6.3)$$

The reader may verify that $\vec{\alpha}_a \cdot \vec{\lambda}^b = \delta_a^b$.

Any root of E_{n+1} can be written as

$$\vec{\alpha} = n_c \vec{\alpha}_n + m\vec{\beta} + \sum_i n_i \vec{\alpha}_i = n_c(0, x, 0) - \vec{\lambda} \qquad (6.4)$$

where $\vec{\lambda} = n_c \vec{\nu} - \sum_i n_i(0, 0, \underline{\alpha}_i) - m(\beta_1, 0, \underline{0})$ is a weight of $SL(2) \otimes SL(n)$. If a representation of $SL(2) \otimes SL(n)$ occurs in the decomposition of the adjoint representation of E_{n+1} its highest weight must occur for some positive integers m, n_i and n_c. We refer to the integer n_c as the level and we can analyse the occurrence of highest weights level by level using the techniques of references [44-46]. Clearly, at level zero i.e $n_c = 0$ we have just the adjoint

representation of $SL(2) \otimes SL(n)$. The result is that the adjoint representation of E_{n+1} contains the adjoint representation of $SL(2) \otimes SL(n)$ at $n_c = 0$ together with the following highest weight representations of $SL(2) \otimes SL(n)$, for $n \leq 7$,

$$
\begin{array}{cccc}
n_c = 1 & n_c = 2 & n_c = 3 & n_c = 4 \\
(\mu_1, \underline{\lambda}^{n-2}) & (0, \underline{\lambda}^{n-4}) & (\mu_1, \underline{\lambda}^{n-6}) & (0, \underline{\lambda}^{n-1})
\end{array} \quad . \tag{6.5}
$$

The weights in the adjoint representation of E_{n+1} are therefore given by

$$
\begin{aligned}
([\beta_1], 0, \underline{0}) \,, & (0, 0, [\underline{\alpha}_1 + \ldots + \underline{\alpha}_{n_1}]) \,, \ ([\mu_1], x, [\underline{\lambda}^{n-2}]) \,, \\
& (0, 2x, [\underline{\lambda}^{n-4}]) \,, ([\mu_1], 3x, [\underline{\lambda}^{n-6}]) \,, (0, 4x, [\underline{\lambda}^{n-1}])
\end{aligned} \tag{6.6}
$$

The first two are simply the weights of the adjoint representation of $SL(2) \otimes SL(n)$. Here $[\bullet]$ denotes the weights in the root string of the $SL(2)$ or $SL(n)$ representation with highest weight $\bullet$. The reader may verify that one does indeed get the correct number of states for the adjoint representation of E_{n+1} for $n = 3, \ldots 7$.

We will need to decompose representations of E_{n+1} other than the adjoint representation. To do this we use the technique of references [43,47]. If one wants to consider the representation of E_{n+1} associated with highest weight $\vec{\lambda}^a$ we add a node to the E_{n+1} Dynkin diagram which is connected to the node a by a single line, thereby constructing the Dynkin diagram for an enlarged algebra of rank $n + 2$. We will denote this additional node by $\star$ and introduce a corresponding level $n_\star$. Deleting this node we recover the E_{n+1} Dynkin diagram. The commutation relations of this new rank $n + 2$ algebra respect the level so the commutation relation between E_{n+1} generators (level zero) and level one gives level one. As such the level one generators form a representation of E_{n+1}; it is in fact the representation with highest weight $\vec{\lambda}^a$. Thus we find the decomposition of the $\vec{\lambda}^a$ representation of E_{n+1} into representations of $SL(2) \otimes SL(n)$ by decomposing the adjoint representation of the enlarged algebra, keeping only contributions with level $n_\star = 1$ and dropping the additional root $\star$.

Alternatively, but this is a more time consuming construction, one can simply start with the highest weight $\vec{\mu}^1$ and lower it with simple roots to construct the entire root string. Given any weight $\vec{\mu}^i$ in the root string and simple root $\vec{\alpha}$, one finds that $\vec{\mu}^i - \vec{\alpha}$ is also in the root string provided that $\vec{\mu}^i \cdot \vec{\alpha} > 0$. In particular the elements of the root string are of the form $\vec{\mu}^i = \vec{\mu}^1 - \sum n_{\vec{\alpha}} \vec{\alpha}$ where $n_{\vec{\alpha}}$ are non-negative integers.

We are most interested in the fundamental representation with highest weight $\vec{\lambda}^{n+1}$ of E_{n+1} as this is the representation from which we construct the automorphic forms that are relevant to string theory [48]. The weights $\vec{\lambda}$ appearing in this representation can be written in the form

$$
[\vec{\lambda}^{n+1}] = \left([\mu], \frac{1}{2x} - n_c x, [\underline{\lambda}] \right) \,. \tag{6.7}
$$

Here $(\mu, \underline{\lambda})$ is the highest weight of the $SL(2) \otimes SL(n)$ representation. We define an ordering of the weights as follows, $\vec{\mu}_i > \vec{\mu}_j$ if the first non-zero component of $\vec{\mu}_i - \vec{\mu}_j$ in the ordered basis $(0, 1, \underline{0})$, $(1, 0, \underline{0})$ and $(0, 0, \underline{\mu})$ is positive. In particular the weights are ordered in terms of increasing level n_c and then with respect to their $SL(2)$ weights and finally with respect to their $SL(n)$ weights. This ordering coincides with that found directly by using the positivity of the roots of the E_{n+1} algebra.

One finds that the weights in the $\vec{\lambda}^{n+1}$ representation of E_{n+1} are given by

$$
\begin{aligned}
&([\mu_1], \frac{1}{2x}, \underline{0}) \,, (0, \frac{1}{2x} - x, [\underline{\lambda}^{n-2}]) \,, ([\mu_1], \frac{1}{2x} - 2x, [\underline{\lambda}^{n-4}]) \,, \\
&(0, \frac{1}{2x} - 3x, [\underline{\lambda}^{n-1}] + [\underline{\lambda}^{n-5}]) \,, (0, \frac{1}{2x} - 3x, [\underline{\lambda}^{n-6}]) \,, ([\beta_1], \frac{1}{2x} - 3x, [\underline{\lambda}^{n-6}]) \,, \\
&([\mu_1], \frac{1}{2x} - 4x, [\underline{\lambda}^{n-1} + \underline{\lambda}^{n-7}]) \,, ([\mu_1], \frac{1}{2x} - 4x, [\underline{\lambda}^{n-6} + \underline{\lambda}^{n-2}]) \,, \\
&(0, \frac{1}{2x} - 5x, [\underline{\lambda}^{n-4} + \underline{\lambda}^{n-6}]) \,, (0, \frac{1}{2x} - 5x, [\underline{\lambda}^{n-1} + \underline{\lambda}^{n-2} + \underline{\lambda}^{n-7}]) \,, \\
&([\beta_1], \frac{1}{2x} - 5x, [\underline{\lambda}^{n-3} + \underline{\lambda}^{n-7}]) \,, ([\mu_1], \frac{1}{2x} - 6x, [\underline{\lambda}^{n-5} + \underline{\lambda}^{n-7}]) \,, \\
&([\mu_1], \frac{1}{2x} - 6x, [2\underline{\lambda}^{n-6}]) \,, ([\mu_1], \frac{1}{2x} - 6x, [\underline{\lambda}^{n-1} + \underline{\lambda}^{n-4} + \underline{\lambda}^{n-7}]) \,, \dots
\end{aligned}
\tag{6.8}
$$

For small values of n many contributions vanish as one has too many anti-symmetrised indices. In particular the $\underline{\lambda}^{n-j}$ representation corresponds to totally anti-symmetric tensors $T^{i_1 \dots i_j}$. The reader may like to verify that one has the correct count of states for the 5,10,27, and 133-dimensional representations of $SL(5)$, $SO(5,5)$, E_6 and E_7 respectively. To find the 3875 dimensional representation of E_8 one must go further in the analysis.

To continue we label the contributions appearing in (6.8) by $\alpha = (n_c, i)$ where the index i labels the different $SL(n)$ representations at the same level n_c. Note that we will explicitly write out each separate state in the $SL(2)$ representation but not each state in the $SL(n)$ representation. We choose our labels so that they increase in accord with the ordering of the weights introduced above. We denote the number of states in the block labeled α by d_α and define $a_\alpha = \sum_{\beta < \alpha} d_\beta$ as well as $b_\alpha = \sum_{\beta < \alpha} n_c(\beta) d_\beta$. The blocks and their labels are explicitly given below for $SL(n+2)$, $SO(5,5)$, E_6 and E_7 in tables 1,2,3,4 respectively.

Substituting the above weights into the general formula for the perturbative contribution of equation (3.19) and using that $\vec{\phi} = (\phi, \rho, \underline{\phi})$ we find that this contribution can be written as

$$
\Phi_p = \sum_{n_c, i} \sum_{k=a_\alpha+1}^{a_{\alpha+1}} E_k N_\alpha(\phi) e^{-\frac{s}{\sqrt{2}x}\rho} e^{(2n_c s - n_c a_\alpha + b_\alpha)\frac{x}{\sqrt{2}}\rho} P_k(\underline{\lambda}, \underline{\phi}) \,,
\tag{6.9}
$$

where

$$
N_\alpha = 1 \,,
\tag{6.10}
$$

if α corresponds to the singlet representation of $SL(2)$,

$$N_\alpha = \begin{cases} e^{-\phi(s-\frac{a_\alpha}{2})} & \text{if weight } \mu_1 \\ e^{\phi(s-\frac{a_{\alpha+1}}{2})} & \text{if weight } -\mu_1 \end{cases} , \tag{6.11}$$

for the doublet representation of $SL(2)$ and

$$N_\alpha = \begin{cases} e^{-\phi(2s-a_\alpha)} & \text{if weight } 2\mu_1 \\ e^{-\phi d_{\alpha-1}} & \text{if weight } 0 \\ e^{\phi(2s-a_{\alpha+1})} & \text{if weight } -2\mu_1 \end{cases} , \tag{6.12}$$

for the triplet representation of $SL(2)$. Furthermore we defined

$$E_k = 2\pi^{\frac{k-1}{2}}\zeta(2s-k+1)\Gamma(s-\frac{k-1}{2})/\Gamma(s) , \tag{6.13}$$

and

$$P_k(\underline{\lambda},\underline{\phi}) = e^{-\sqrt{2}((s-\frac{k-1}{2})[\underline{\lambda}]_{k-a_\alpha}+\frac{1}{2}([\underline{\lambda}]_1+...+[\underline{\lambda}]_{k-1-a_\alpha}))\cdot\underline{\phi}} , \tag{6.14}$$

where $[\underline{\lambda}]_r$ is the r-th weight in the root string with highest $SL(n)$ weight $\underline{\lambda}$.

6.1 $SL(m+2) \to SL(2) \times SL(m)$

Let us start by considering the case of $SL(m+2) \to SL(2) \times SL(m)$ which provides a simple example of the methods we explained above. The Dynkin diagram of $SL(m+2)$ before and after deleting the m-th node is given in Figure 3.
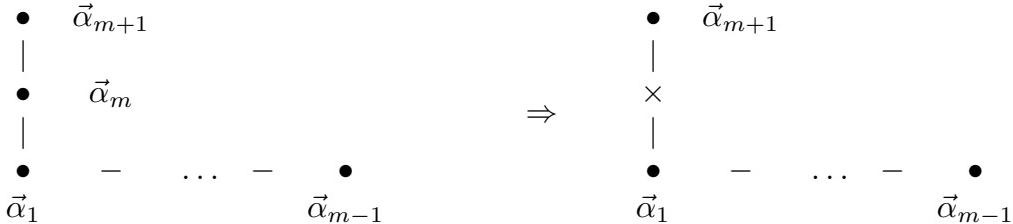
FIGURE 3

For $m > 1$ the simple roots can be written as

$$\vec{\alpha}_{m+1} = (\sqrt{2},0,\underline{0}) \qquad \vec{\alpha}_m = (-\frac{1}{\sqrt{2}},x,-\underline{\lambda}^1) \qquad \vec{\alpha}_i = (0,0,\underline{\alpha}_i) , \tag{6.15}$$

where $x = \sqrt{(m+2)/2m}$ and the fundamental weights are

$$\vec{\lambda}^{m+1} = (\frac{1}{\sqrt{2}},\frac{1}{2x},\underline{0}) \qquad \vec{\lambda}^m = (0,\frac{1}{x},\underline{0}) \qquad \vec{\lambda}^i = (0,\frac{1}{x}\underline{\lambda}^i \cdot \lambda^1,\underline{\lambda}^i) . \tag{6.16}$$

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For $m = 1$, i.e. $SL(3)$, we have the roots $\vec{\alpha}_{m+1} = (\sqrt{2}, 0)$ and $\vec{\alpha}_m = (-\frac{1}{\sqrt{2}}, x)$ and weights $\vec{\lambda}^{m+1} = (\frac{1}{\sqrt{2}}, \frac{1}{2x})$, $\vec{\lambda}^m = (0, \frac{1}{x})$.

It is not difficult to see that the weights of the $\vec{\lambda}^{m+1}$ representation are

$$([\mu^1], \frac{1}{2x}, \underline{0}), (0, \frac{1}{2x} - x, [\underline{\lambda}^1]) , \tag{6.17}$$

where the first entry is the $SL(2)$ weight and the last the $SL(m)$ weight. Thus we see that $\mathbf{m + 2} \rightarrow (\mathbf{2}, \mathbf{1})_\mathbf{0} \oplus (\mathbf{1}, \mathbf{m})_\mathbf{1}$, where the subscript indicates the level. To compute the perturbative part of the automorphic form we can now simply use the formula (6.9). The various parameters needed that are computed from the group decomposition are listed in Table 1.

	(0,1)	(0,2)	(1,1)	
$SL(m)$ rep.	$\underline{0}$	$\underline{0}$	$\underline{\lambda}^1$	
$SL(2)$ weight	μ	$-\mu$	0	
d_α	1	1	n	
a_α	0	1	2	$n+2$
b_α	0	0	0	n

Table 1: $SL(m+2) \rightarrow SL(2) \oplus SL(m)$

Reading off the data from Table 1 and using the formula (6.9) we find

$$\Phi_p = e^{-s\phi} e^{-\frac{s}{\sqrt{2}x}\rho} E_1 + e^{(s-1)\phi} e^{-\frac{s}{\sqrt{2}x}\rho} E_2 + e^{-\frac{s}{\sqrt{2}x}\rho} e^{(2s-2)\frac{x}{\sqrt{2}}\rho} \sum_{k=3}^{m+2} E_k P_k(\underline{\lambda}^1, \underline{\phi}) , \tag{6.18}$$

where $x = \sqrt{(m+2)/2m}$.

For $m = 1$ we find the $\mathbf{3}$ of $SL(3)$ which arises in String Theory as a special case of $SL(3) \times SL(2)$. In addition for $m = 3$ we find the $\mathbf{5}$ of $SL(5)$ which also arises in String Theory. We note that taking $n = 2$ in equation (6.2) gives $x = \sqrt{3/2}$ which agrees with $x = \sqrt{(m+2)/2m}$ for $m = 1$. Indeed one can see that figure 3 agrees with figures 1 and 2 for these two cases, namely $n = 2$ and $n = 3$, although for $n = 2$ one finds an addition node in figures 1 and 2 corresponding to the extra $SL(2)$ in $SL(3) \times SL(2)$. For $m \neq 1, 3$ we find more general possibilities than those which arise in string theory.

6.2 $n = 4, E_5 = SO(5,5)$

	(0,1)	(0,2)	(1,1)	(2,1)	(2,2)	
$SL(4)$ rep.	$\underline{0}$	$\underline{0}$	$\underline{\lambda}^2$	$\underline{0}$	$\underline{0}$	
$SL(2)$ weight	μ	$-\mu$	0	μ	$-\mu$	
d_α	1	1	6	1	1	
a_α	0	1	2	8	9	10
b_α	0	0	0	6	8	10

Table 2: $SO(5,5) \to SL(2) \oplus SL(4)$

$$\Phi_p = e^{-s\phi}e^{-\frac{s}{\sqrt{2}x}\rho}E_1 + e^{(s-1)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}E_2 + e^{-\frac{s}{\sqrt{2}x}\rho}e^{(2s-2)\frac{x}{\sqrt{2}}\rho}\sum_{k=3}^{8}E_k P_k(\underline{\lambda}^2,\underline{\phi})$$
$$+ e^{-(s-4)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(4s-10)\frac{x}{\sqrt{2}}\rho}E_9 P_9(\underline{0},\underline{\phi}) + e^{(s-5)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(4s-10)\frac{x}{\sqrt{2}}\rho}E_{10}P_{10}(\underline{0},\underline{\phi}) \ . \tag{6.19}$$

Substituting $x = 1/\sqrt{2}$ gives

$$\Phi_p = e^{-s\phi}e^{-s\rho}E_1 + e^{(s-1)\phi}e^{-s\rho}E_2 + e^{-\rho}\sum_{k=3}^{8}E_k P_k(\underline{\lambda}^2,\underline{\phi})$$
$$+ e^{-(s-4)\phi}e^{(s-5)\rho}E_9 P_9(\underline{0},\underline{\phi}) + e^{(s-5)\phi}e^{(s-5)\rho}E_{10}P_{10}(\underline{0},\underline{\phi}) \ . \tag{6.20}$$

6.3 $n = 5, E_6$

	(0,1)	(0,2)	(1,0)	(2,1)	(2,2)	(3,1)	
$SL(5)$ rep.	$\underline{0}$	$\underline{0}$	$\underline{\lambda}^3$	$\underline{\lambda}^1$	$\underline{\lambda}^1$	$\underline{\lambda}^4$	
$SL(2)$ weight	μ	$-\mu$	0	μ	$-\mu$	0	
d_α	1	1	10	5	5	5	
a_α	0	1	2	12	17	22	27
b_α	0	0	0	10	20	30	45

Table 3: $E_6 \to SL(2) \oplus SL(5)$

$$\Phi_p = e^{-s\phi}e^{-\frac{s}{\sqrt{2}x}\rho}E_1 + e^{(s-1)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}E_2$$

$$+ e^{-\frac{s}{\sqrt{2}x}\rho}e^{(2s-2)\frac{x}{\sqrt{2}}\rho}\sum_{k=3}^{12}E_kP_k(\underline{\lambda}^3,\underline{\phi}) + e^{-(s-6)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(4s-14)\frac{x}{\sqrt{2}}\rho}\sum_{k=13}^{17}E_kP_k(\underline{\lambda}^1,\underline{\phi})$$

$$+ e^{(s-11)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(4s-14)\frac{x}{\sqrt{2}}\rho}\sum_{k=18}^{22}E_kP_k(\underline{\lambda}^1,\underline{\phi}) + e^{-\frac{s}{\sqrt{2}x}\rho}e^{(6s-36)\frac{x}{\sqrt{2}}\rho}\sum_{k=23}^{27}E_kP_k(\underline{\lambda}^4,\underline{\phi}) \ .$$

$$(6.21)$$

Substituting $x = \sqrt{3/10}$ gives

$$\Phi_s = e^{-s\phi}e^{-s\sqrt{\frac{5}{3}}\rho}E_1 + e^{(s-1)\phi}e^{-s\sqrt{\frac{5}{3}}\rho}E_2$$

$$+ e^{-(2s+3)\rho/\sqrt{15}}\sum_{k=3}^{12}E_kP_k(\underline{\lambda}^3,\underline{\phi}) + e^{-(s-6)\phi}e^{(s-21)\rho/\sqrt{15}}\sum_{k=13}^{17}E_kP_k(\underline{\lambda}^1,\underline{\phi})$$

$$+ e^{(s-11)\phi}e^{-(s-6)\phi}e^{(s-21)\rho/\sqrt{15}}\sum_{k=18}^{22}E_kP_k(\underline{\lambda}^1,\underline{\phi}) + e^{(4s-54)\rho/\sqrt{15}}\sum_{k=23}^{27}E_kP_k(\underline{\lambda}^4,\underline{\phi}) \ .$$

$$(6.22)$$

6.4 $n = 6, E_7$

	(0,1)	(0,2)	(1,0)	(2,1)	(2,2)	(3,0)	(3,1)
$SL(6)$ rep.	$\underline{0}$	$\underline{0}$	$\underline{\lambda}^4$	$\underline{\lambda}^2$	$\underline{\lambda}^2$	$\underline{0}$	$\underline{\theta}_{\geq}$
$SL(2)$ weight	μ	$-\mu$	0	μ	$-\mu$	2μ	0
d_α	1	1	15	15	15	1	20
a_α	0	1	2	17	32	47	48
b_α	0	0	0	15	45	75	78

(3,2)	(3,3)	(3,4)	(3,5)	(4,1)	(4,2)	(5,1)	(6,1)	(6,2)	
$\underline{0}$	$\underline{0}$	$\underline{\theta}_{<}$	$\underline{0}$	$\underline{\lambda}^4$	$\underline{\lambda}^4$	$\underline{\lambda}^2$	$\underline{0}$	$\underline{0}$	
0	0	0	-2μ	μ	$-\mu$	0	μ	$-\mu$	
1	1	15	1	15	15	15	1	1	
68	69	70	85	86	101	116	131	132	133
138	141	144	189	192	252	312	387	393	399

Table 3: $E_7 \rightarrow SL(2) \oplus SL(6)$. Note that $\underline{\theta} = \underline{\lambda}^1 + \underline{\lambda}^5$ is the highest weight of the adjoint representation of $SL(6)$, $\underline{\theta}_{\geq}$ denotes the non-negative roots of $SL(6)$ and $\underline{\theta}_{<}$ the negative roots.

$$\Phi_p = e^{-s\phi}e^{-\frac{s}{\sqrt{2}x}\rho}E_1 + e^{(s-1)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}E_2$$

$$+ e^{-\frac{s}{\sqrt{2}x}\rho}e^{(2s-2)\frac{x}{\sqrt{2}}\rho}\sum_{k=3}^{17}E_kP_k(\underline{\lambda}^4,\underline{\phi}) + e^{-(s-17/2)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(4s-19)\frac{x}{\sqrt{2}}\rho}\sum_{k=18}^{32}E_kP_k(\underline{\lambda}^2,\underline{\phi})$$

$$+ e^{(s-47/2)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(4s-19)\frac{x}{\sqrt{2}}\rho}\sum_{k=33}^{47}E_kP_k(\underline{\lambda}^2,\underline{\phi})$$

$$+ e^{-(2s-47)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(6s-66)\frac{x}{\sqrt{2}}\rho}E_{48}P_{48}(\underline{0},\underline{\phi})$$

$$+ e^{-\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(6s-66)\frac{x}{\sqrt{2}}\rho}\sum_{k=49}^{68}E_kP_k(\underline{\lambda}^1+\underline{\lambda}^5,\underline{\phi})$$

$$+ e^{-\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(6s-66)\frac{x}{\sqrt{2}}\rho}E_{69}e^{-\frac{1}{\sqrt{2}}\sum_{\underline{\alpha}>0}\underline{\alpha}\cdot\underline{\phi}} + e^{-\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(6s-66)\frac{x}{\sqrt{2}}\rho}E_{70}e^{-\frac{1}{\sqrt{2}}\sum_{\underline{\alpha}>0}\underline{\alpha}\cdot\underline{\phi}}$$

$$+ e^{-\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(6s-66)\frac{x}{\sqrt{2}}\rho}\sum_{k=71}^{85}E_kP_k(\underline{\lambda}^1+\underline{\lambda}^5,\underline{\phi})$$

$$+ e^{(2s-86)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(6s-66)\frac{x}{\sqrt{2}}\rho}E_{86}P_{86}(\underline{0},\underline{\phi})$$

$$+ e^{-(s-86/2)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(8s-152)\frac{x}{\sqrt{2}}\rho}\sum_{k=87}^{101}E_kP_k(\underline{\lambda}^4,\underline{\phi})$$

$$+ e^{(s-116/2)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(8s-152)\frac{x}{\sqrt{2}}\rho}\sum_{k=102}^{116}E_kP_k(\underline{\lambda}^4,\underline{\phi})$$

$$+ e^{-\frac{s}{\sqrt{2}x}\rho}e^{(10s-268)\frac{x}{\sqrt{2}}\rho}\sum_{k=117}^{131}E_kP_k(\underline{\lambda}^2,\underline{\phi})$$

$$+ e^{-(s-131/2)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(12s-399)\frac{x}{\sqrt{2}}\rho}E_{132}P_{132}(\underline{0},\underline{\phi})$$

$$+ e^{(s-133/2)\phi}e^{-\frac{s}{\sqrt{2}x}\rho}e^{(12s-399)\frac{x}{\sqrt{2}}\rho}E_{133}P_{133}(\underline{0},\underline{\phi}) \ .$$

(6.23)

Note that some care in using (6.9) is required here as the adjoint representation of $SL(n)$

appearing at level 3 is split by some $SL(n)$ singlet states. Substituting $x = \sqrt{1/6}$ gives

$$
\begin{aligned}
\Phi_s &= e^{-s\phi}e^{-s\sqrt{3}\rho}E_1 + e^{(s-1)\phi}e^{-s\sqrt{3}\rho}E_2 \\
&+ e^{-(2s+1)\rho/\sqrt{3}}\sum_{k=3}^{17} E_k P_k(\underline{\lambda}^4, \underline{\phi}) + e^{-(s-17/2)\phi}e^{-(s+19/2)\rho/\sqrt{3}}\sum_{k=18}^{32} E_k P_k(\underline{\lambda}^2, \underline{\phi}) \\
&+ e^{(s-47/2)\phi}e^{-(s+19/2)\rho/\sqrt{3}}\sum_{k=33}^{47} E_k P_k(\underline{\lambda}^2, \underline{\phi}) + e^{-(2s-47)\phi}e^{-11\sqrt{3}\rho}E_{48}P_{48}(\underline{0}, \underline{\phi}) \\
&+ e^{-\phi}e^{-11\sqrt{3}\rho}\sum_{k=49}^{68} E_k P_k(\underline{\lambda}^1 + \underline{\lambda}^5, \underline{\phi}) + e^{-\phi}e^{-11\sqrt{3}\rho}(E_{69} + E_{70})e^{-\frac{1}{\sqrt{2}}\sum_{\underline{\alpha}>0}\underline{\alpha}\cdot\underline{\phi}} \\
&+ e^{-\phi}e^{-11\sqrt{3}\rho}\sum_{k=71}^{85} E_k P_k(\underline{\lambda}^1 + \underline{\lambda}^5, \underline{\phi}) + e^{(2s-86)\phi}e^{-11\sqrt{3}\rho}E_{86}P_{86}(\underline{0}, \underline{\phi}) \\
&+ e^{-(s-86/2)\phi}e^{(s-76)\rho/\sqrt{3}}\sum_{k=87}^{101} E_k P_k(\underline{\lambda}^4, \underline{\phi}) + e^{(s-116/2)\phi}e^{(s-76)\rho/\sqrt{3}}\sum_{k=102}^{116} E_k P_k(\underline{\lambda}^4, \underline{\phi}) \\
&+ e^{(2s-134)\rho/\sqrt{3}}\sum_{k=117}^{131} E_k P_k(\underline{\lambda}^2, \underline{\phi}) + e^{-(s-131/2)\phi}e^{(3s-399/2)\rho/\sqrt{3}}E_{132}P_{132}(\underline{0}, \underline{\phi}) \\
&+ e^{(s-133/2)\phi}e^{(3s-399/2)\rho/\sqrt{3}}E_{133}P_{133}(\underline{0}, \underline{\phi}) \ .
\end{aligned}
$$

$$(6.24)$$

7. Applications to Perturbative String Theory

We begin with type IIB supergravity in ten-dimensions [4,5,6], viewed as the effective action of the type IIB superstring. This theory contains two scalar fields: the dilaton ϕ which controls the string coupling constant, and a RR axion-like field χ. The metric and scalar part of the lowest order supergravity effect action is, in Einstein frame,

$$
S = \frac{1}{\alpha'^4}\int d^{10}x\sqrt{-g}(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial_\mu\chi\partial^\mu\chi) \ . \tag{7.1}
$$

Our first step is to show that these fields can be identified with the fields in the $SL(2)/SO(2)$ coset representative g in (5.2) that we used above and fix any normalizations. This justifies our use of the same symbols for both the supergravity fields and the coset representative. Let us consider the coset g that we introduced above in (5.2) and consider the Cartan form

$$
g^{-1}\partial_\mu g = -\frac{1}{\sqrt{2}}\partial_\mu\phi H + e^\phi\partial_\mu\chi E_{\beta_1} \ . \tag{7.2}
$$

Under a coset transformation $g \to g_0 g h^{-1}$ we see that $g^{-1}\partial_\mu g \to h g^{-1}\partial_\mu g h^{-1} + h\partial_\mu h^{-1}$.

Since the second term is in the Lie algebra of $H = SO(2)$ we see that

$$\begin{aligned}
\mathcal{P}_\mu &= \frac{1}{2}g^{-1}\partial_\mu g + \frac{1}{2}(g^{-1}\partial_\mu g)^T \\
&= -\frac{1}{\sqrt{2}}\partial_\mu \phi H + \frac{1}{2}e^\phi \partial_\mu \chi (E_{\beta_1} + F_{-\beta_1})
\end{aligned} \tag{7.3}$$

transforms as $\mathcal{P}_\mu \to h\mathcal{P}_\mu h^{-1}$ under $SL(2)$. The action can now be written in the manifestly $SL(2)$ invariant form

$$\begin{aligned}
S &= \frac{1}{\alpha'^4} \int d^{10}x \sqrt{-g}(R - tr(\mathcal{P}_\mu \mathcal{P}^\mu)) \\
&= \frac{1}{\alpha'^4} \int d^{10}x \sqrt{-g}(R - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}e^{2\phi}\partial_\mu \chi \partial^\mu \chi) \ .
\end{aligned} \tag{7.4}$$

In particular this shows that we can identify the scalar ϕ that appears in the coset with the ten-dimensional type IIB dilaton and the RR scalar χ with the scalar field associated to the positive root generator of $SL(2)$ (up to a possible sign $\chi \to -\chi$).

Let us now consider type IIB string theory compactified on T^n. We use the compactification ansatz

$$ds_E^2 = e^{2\alpha\rho}ds_d^2 + e^{2\beta\rho}G_{ij}(dx^i + A^i)(dx^j + A^j) \ , \tag{7.5}$$

where G_{ij} is metric on the internal T^n with unit determinant. The parameters α and β are chosen to ensure that the reduced action is in Einstein frame (assuming one starts with the ten-dimensional Einstein frame) and also that the modulus ρ has a standard kinetic term. This determines

$$\alpha = \frac{1}{4}\sqrt{\frac{n}{8-n}} \qquad \beta = -\frac{1}{4}\sqrt{\frac{8-n}{n}} \ . \tag{7.6}$$

Note that $\alpha = \frac{1}{4\sqrt{2}x}$ and $\beta = -\frac{\sqrt{2}x}{4}$ where x arose in the decomposition of $E_{n+1} \to SL(2) \times SL(n)$ and was determined in (6.2).

The internal metric $G_{ij} = e_i{}^{\overline{k}}e_j{}^{\overline{l}}\delta_{\overline{kl}}$ gives rise to scalar fields in the dimensionally reduced theory. It can be shown that the internal vielbein $e_i{}^{\overline{k}}$ is itself a $SL(n)/SO(n)$ coset representative. It therefore contains $n-1$ scalars $\underline{\phi}$ that are associated to the Cartan subalgebra of $SL(n)$ as well as $n(n-1)/2$ scalars $\chi_{\underline{\alpha}}$ associated to the positive root generators of $SL(n)$. We have the ten-dimensional dilaton ϕ and the volume modulus ρ as well as other scalar fields such as χ and components of the p-form gauge fields in the internal dimensions.

Thus we see that type IIB supergravity compactified on an n torus has an $SL(2) \times SL(n)$ symmetry. In fact one finds precisely the right scalar fields to parameterize an $E_{n+1}/I(E_{n+1})$ coset $g(\xi)$, where $I(E_{n+1})$ is the Cartan involution invariant subgroup and

also the maximally compact subgroup of E_{n+1}. In particular the fields associated to the Cartan subalgebra of E_{n+1} are ϕ, ρ and $\underline{\phi}$ and these can identified with

$$\vec{\phi} = (\phi, \rho, \underline{\phi}) , \tag{7.7}$$

as we did in the previous section. It is a remarkable fact that the entire effective supergravity theory has a E_{n+1} symmetry at lowest order in derivatives.

As is well-known, a discrete E_{n+1} U-duality is conjecture to hold in the full quantum string theory. Therefore the complete low energy effective action containing higher derivative terms must posses a discrete E_{n+1} symmetry. The terms in the effective action are made of powers of the various field strengths, which form representations of E_{n+1}, along with functions of the scalar fields which must be automorphic forms of $E_{n+1}/I(E_{n+1})$. (Note that particular care must be taken for $d/2$-form field strengths which do not generally form E_{n+1} multiplets without also including their electromagnetic duals.) In particular such a term has the form

$$\mathcal{L}_{\mathcal{O}} = \sqrt{-g}\Phi(g)\mathcal{O} , \tag{7.8}$$

where Φ is an automorphic form and

$$\mathcal{O} \sim D^{2\delta} R^{l_R/2}(\mathcal{P})^{l_1}(\mathcal{F}_2)^{l_2}(\mathcal{F}_3)^{l_3}(\mathcal{F}_4)^{l_4} . \tag{7.9}$$

Here R is the Riemann tensor, $\mathcal{P}$ the component of the E_{n+1} Cartan form that is not in the $I(E_{n+1})$ subalgebra and $\mathcal{F}_{p+2} = L(g^{-1})F_{p+2}$ where $F_{p+2} = dA_{p+1}$ is a $(p+2)$-form field strength. Note that $\mathcal{F}_{p+2}$ is constructed to transform under local $I(E_{n+1})$ transformations. Thus all the fields in $\mathcal{O}$ transform in some representation of $I(E_{n+1})$ and U-duality requires that $\mathcal{L}_{\mathcal{O}}$ is $I(E_{n+1})$-invariant.

In ten dimensions string perturbation theory is an expansion in $g_s = e^\phi$ and naturally takes place in the so-called string frame where

$$S = \frac{1}{\alpha'^4} \int d^{10}x \sqrt{-g_S} g_s^{-2} R_S + \dots \tag{7.10}$$

Upon reduction to $d = 10 - n$ dimensions we find

$$S = \frac{1}{\alpha'^4} \int d^d x \sqrt{-g_S} V_n g_s^{-2} R_S + \dots = \frac{1}{\alpha'^{\frac{d-2}{2}}} \int d^d x \sqrt{-g_S} g_d^{-2} R_S + \dots \tag{7.11}$$

where V_n is the volume of T^n (in string frame and string units) and $g_d = \alpha'^{\frac{n}{4}} g_s / \sqrt{V_n}$ is the effective coupling constant in d-dimensions. Using our reduction ansatz (7.5) we find

$$V_n = e^{n\phi/4 + n\beta\rho} \qquad g_d = e^{\frac{8-n}{8}\phi - n\beta\rho/2} . \tag{7.12}$$

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Since the dilaton ϕ is not invariant under a duality transformation the duality structure of the effective action is most manifest in Einstein frame which, in d-dimensions, is related to the string frame by

$$(g_E)_{\mu\nu} = g_d^{-\frac{4}{d-2}}(g_S)_{\mu\nu} \ . \tag{7.13}$$

When we rescale to string frame the term (7.8) becomes

$$\mathcal{L}_\mathcal{O} = \sqrt{-g_S}g_d^{\frac{4\Delta-2d}{d-2}}\Phi\mathcal{O}_S \ , \tag{7.14}$$

where $\Delta = \delta + \frac{l_R}{2} + \frac{l_1}{2} + 2\frac{l_2}{2} + 3\frac{l_3}{2} + 4\frac{l_4}{2}$ counts the number of inverse metrics that contained in $\mathcal{L}_\mathcal{O}$ and $\mathcal{O}_S$ denotes $\mathcal{O}$ with variables transformed to the string frame. The perturbative terms that arise in Φ need to be consistent with perturbation theory. This requires that, in d-dimensional string frame, each term must be of the form g_d^{2g-2} where $g = 0, 1, 2, ..$ is the genus. Therefore we require that

$$g_d^{\frac{4\Delta-2d}{d-2}}\Phi_p \ , \tag{7.15}$$

only contains terms of the form g_d^{2g-2}, $g = 0, 1, 2, ...$, which we can identify as arising from perturbation theory.

Let us examine the automorphic forms of E_{n+1} constructed from representations whose highest weight is $\vec{\lambda}^{n+1}$. The perturbative terms where given in the previous section. To make contact with String Theory here we are interested in their dependence on g_d and V_n. Using (6.3) we see that the first term ($k = 1$) in Φ_p is

$$2\zeta(2s)e^{-s\vec{\phi}\cdot\vec{\lambda}^{n+1}} = 2\zeta(2s)g_d^{-\frac{8s}{8-n}} \ . \tag{7.16}$$

Demanding that this comes from lowest order in perturbation theory gives

$$-\frac{8s}{8-n} + \frac{4\Delta-2d}{d-2} = -2 \ , \tag{7.17}$$

and hence we deduce that

$$s = (\Delta-1)/2 \ . \tag{7.18}$$

Before we write the complete perturbative parts of the automorphic forms in terms of g_d and V_n it is helpful to make the following observation. The formula (3.19) the k-th term in the perturbative part involves the weight $\vec{w}_k = -\sqrt{2}(s - \frac{k-1}{2})\vec{\mu}^k - \frac{1}{\sqrt{2}}(\vec{\mu}^1 + ... + \vec{\mu}^{k-1})$. Thus that the difference between the weights of any two consecutive terms is

$$\begin{aligned}
\vec{w}_k - \vec{w}_{k+1} &= -\sqrt{2}(s - \frac{k-1}{2})\vec{\mu}^k + \sqrt{2}(s - \frac{k}{2})\vec{\mu}^{k+1} + \frac{1}{\sqrt{2}}\vec{\mu}^k \\
&= -\sqrt{2}(s - k/2)(\vec{\mu}^k - \vec{\mu}^{k+1}) \ .
\end{aligned} \tag{7.19}$$

In particular we need to evaluate $e^{(\vec{w}_k - \vec{w}_{k+1}) \cdot \vec{\phi}}$. Now $\vec{\mu}^k - \vec{\mu}^{k+1} = q^i \vec{\alpha}_i$ is a positive element of the root lattice (note that the q^i need not all be positive). From (6.1) we see that

$$
\begin{aligned}
e^{\sqrt{2} \vec{\alpha}_{n+1} \cdot \vec{\phi}} &= g_d^2 V_n \\
e^{\sqrt{2} \vec{\alpha}_n \cdot \vec{\phi}} &= V_n^{-\frac{4}{n}} e^{-\sqrt{2} \underline{\lambda}^{n-2} \cdot \underline{\phi}} \\
e^{\sqrt{2} \vec{\alpha}_j \cdot \vec{\phi}} &= e^{\sqrt{2} \underline{\alpha}_j \cdot \underline{\phi}} \ .
\end{aligned}
\tag{7.20}
$$

Thus the power of g_d only changes as we work our way down the root string if $\vec{\mu}^k - \vec{\mu}^{k+1}$ contains $\vec{\alpha}_{n+1}$, $e.g.$ if the $SL(2)$ weight within a given $SL(2)$ representation is lowered. In this case we see that the power of g_d changes by an integer multiple of $2s - k$. In addition we see that the volume dependence only changes if we either change the $SL(2)$ weight or the level n_c, $i.e.$ it is constant for any given $SL(n)$ representation.

Let us now write down the perturbative automorphic forms of E_{n+1} constructed from the $\vec{\lambda}^{n+1}$ representation. To convert the previous formulae it is useful to observe that

$$
e^{\phi} = g_d V_n^{\frac{1}{2}} \qquad e^{\frac{\rho}{\sqrt{2x}}} = V_n^{-\frac{1}{2}} g_d^{\frac{n}{8-n}} \qquad e^{\frac{x}{\sqrt{2}} \rho} = g_d^{\frac{1}{2}} V_n^{-\frac{8-n}{4n}} \ .
\tag{7.21}
$$

Our results for the explicit expressions for Φ_p that we gave in section 6 are listed below. We have written out the expressions with the expectation that the leading order term is tree-level in string theory and used the fact that $P_k(\underline{0}, \underline{\phi}) = 1$. That is we have written the power of g_d required to convert the result to string frame in front of the automorphic form and so the result in string frame is the integral $\int d^d x \sqrt{-g}$ times the expressions given below.

7.1 $d = 10, E_1 = SL(2)$

From equation (5.11) we find the perturbative result to be

$$
g_d^{s-2} \Phi_p = g_d^{-2} E_1 + g_d^{2s-3} E_2 \ ,
\tag{7.22}
$$

This is the ten dimensional IIB string theory and this automorphic form and its relations to string theory has been much studied [19-26]. We include it here for completeness and to illustrate the method we are employing. We see that the automorphic form has all the features that we would expect. In particular the perturbative part is a power series expansion in $g_s = e^{\phi}$ corresponding to a tree ($g = 0$) and $g = s - \frac{1}{2}$ correction and is independent of χ. Thus if we take $s = (\Delta - 1)/2$ then we find contributions at genus $g = 0, \Delta/2 - 1$. We can check that for $s = \frac{3}{2}$, that is R^4 the power of g_s required to go to string frame is $g_s^{-\frac{1}{2}}$ and this is indeed the factor multiplying the automorphic form above. In addition the non-perturbative part contains all the χ-dependence and,

indeed $K_{s-1/2}(2\pi m_2|m_1|e^{-\phi})$ is exponentially suppressed as $e^{\phi} \to 0$. Expanding the Bessel function of equation (5.12) we find the non-perturbative contribution is given by [23]

$$\Phi_{np} = \frac{\pi^s}{\Gamma(s)} \sum_{p\neq 0} \sum_{m\neq 0} e^{2\pi i(pm\chi + i|pm|g_s^{-1})} \left|\frac{p}{m}\right|^s \frac{1}{|p|} \sum_{k=0}^{\infty} \frac{g_s^k}{(4\pi|mp|)^k} \frac{\Gamma(s+k)}{\Gamma(k+1)\Gamma(s-k)} . \quad (7.23)$$

We note the typical non-perturbative behaviour $e^{-\frac{2\pi|pm|}{g_s}}$.

7.2 $d = 8, E_3 = SL(3) \times SL(2)$

Using equation (7.21) to convert equation (6.18) to string theory quantities we find the perturbative part of the $SL(3)$ automorphic function is given by

$$g_d^{4s/3-2}\Phi_p = g_d^{-2}E_1 + g_d^{2s-3}V_n^{s-1/2}E_2 + g_d^{2s-3}V_n^{-s+3/2}E_3 . \quad (7.24)$$

Since this is just an $SL(3)$ automorphic form the result applies to terms when there is no $SL(2)$ automorphic form present, but one can also use it to give the $SL(3)$ contribution if the latter is present. Taking $s = (\Delta - 1)/2$ then we find contributions at genus $g = 0, \Delta/2 - 1$.

This automorphic form has been conjectured to arise in String Theory as a coefficient of the R^4 term in 8 dimensions with $s = 3/2$ [27] where a comparison with string theory results was carried out. In this case it is divergent. The regularization does not affect Φ_{np} or the first two terms in Φ_p however the third term is divergent and, following the discussion in section 4, we find it is, in the $s \to 3/2$ limit

$$2\pi \frac{\Gamma(s-1)}{\Gamma(s)}\zeta(2s-2)e^{(2s-3)\rho/\sqrt{3}} \longrightarrow \frac{2\pi}{\epsilon} + 4\pi(\gamma - 1) + 4\pi\rho/\sqrt{3} + \mathcal{O}(\epsilon) , \quad (7.25)$$

where γ is the Euler constant. A suitable renormalized automorphic form is obtained by subtracting off the $\frac{2\pi}{\epsilon} + 4\pi(\gamma - 1)$ factor.

This coefficient of the D^4R^4 terms has been conjectured to be an automorphic form with $s = \frac{5}{2}$ and some checks with string theory results have been carried out [27,28].

The non-perturbative part of the $SL(3)$ can be found from equation (3.21) using the change to physical variables of equation (7.21). We can write the result as $\Phi_{np} = \Phi_{np}^{(1)} + \Phi_{np}^{(2)}$ where

$$\Phi_{np}^{(1)} = \frac{2\pi^s}{\Gamma(s)}(V_n)^{\frac{s}{2}-\frac{1}{4}}(g_d)^{-s-\frac{1}{2}} \sum_{p\neq 0} \sum_{(m_2,m_3)\neq(0,0)} \left|\frac{\nu_2}{p^2}\right|^{\frac{1}{4}-\frac{s}{2}} e^{2\pi ip\tilde{\chi}_1} K_{s-\frac{1}{2}}(2\pi|p||\nu_2|^{\frac{1}{2}}g_d^{-1}V_n^{-\frac{1}{2}})$$

$$\Phi_{np}^{(2)} = \frac{2\pi^s}{\Gamma(s)}(V_n)^{\frac{1}{2}}(g_d)^{\frac{2s}{3}-1} \sum_{p\neq 0} \sum_{m_3,m_3\neq 0} \left|\frac{m_3}{p^2}\right|^{1-s} e^{2\pi ipm_3\chi_{\tilde{\alpha}_2}} K_{s-1}(2\pi|pm_3|V_n^{-2}) .$$

$$(7.26)$$

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In these equations $\nu_2 = (m_2 - \chi_{\bar{\alpha}_2} m_3)^2 + m_3^2 V_n^2$ and $\tilde{\chi}_1 = m_2 \chi_{\bar{\alpha}_1} + m_3 \chi_{\bar{\alpha}_1 + \bar{\alpha}_2} - \frac{1}{2} \chi_{\bar{\alpha}_1} \chi_{\bar{\alpha}_2}$. This result essentially agrees with that of reference [27], although not in every detail.

It is instructive to use equation (A.4) to carry out the $g_d \to 0$ expansion of the non-perturbative result. We find that

$$
\Phi_{np}^{(1)} = \frac{2\pi^s}{\Gamma(s)} V_n^{\frac{s}{2} + \frac{1}{4}} g_d^{-s + \frac{1}{2}} \sum_{p \neq 0} \sum_{(m_2, m_3) \neq (0,0)} \left| \frac{\nu_2}{p^2} \right|^{\frac{1}{4} - \frac{s}{2}} \frac{e^{2\pi i p \tilde{\chi}_1}}{|p\nu_2^{\frac{1}{2}}|} e^{-2\pi |p\nu_2| g_d^{-1} V_n^{-\frac{1}{2}}}
$$
$$
\times \sum_{k=0}^{\infty} \left(\frac{V_n^{\frac{1}{2}} g_d}{4\pi |p\nu_2^{\frac{1}{2}}|} \right)^k \frac{\Gamma(s+k)}{\Gamma(k+1)\Gamma(s-k)}
$$
(7.27)

while

$$
\Phi_{np}^{(2)} = \frac{2\pi^s}{\Gamma(s)} V_n^{\frac{3}{2}} g_d^{\frac{2s}{3} - 1} \sum_{p \neq 0} \sum_{m_3 \neq 0} \left| \frac{m_3}{p^2} \right|^{1-s} \frac{e^{2\pi i (p m_3 \chi_{\bar{\alpha}_2} + i|p m_3| V_n^{-2})}}{\sqrt{|p m_3|}}
$$
$$
\times \sum_{k=0}^{\infty} \left(\frac{V_n^2}{4\pi |p m_3|} \right)^k \frac{\Gamma(s + k - \frac{1}{2})}{\Gamma(k+1)\Gamma(s - k - \frac{1}{2})} .
$$
(7.28)

We note that the series of terms in the second term always terminates for half integer s and for $s = \frac{3}{2}$, that is for R^4, only the first term survives. Using equation (7.21) to convert to string variables we find that the second term for $s = \frac{3}{2}$ can be written as

$$
\Phi_{np}^{(2)} = 2\pi V_n^{\frac{3}{2}} \sum_{p \neq 0} \sum_{m \neq 0} \frac{1}{|m|} e^{2\pi i (p m \tilde{\chi}_{\bar{\alpha}_2} + i|p m| V_n^{-2})}
$$
$$
= 4\pi V_n^{\frac{3}{2}} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \mathrm{Re} \left(e^{2\pi i m p \mathcal{T}} \right)
$$
(7.29)
$$
= -8\pi V_n^{\frac{3}{2}} \sum_{\hat{p}=1}^{\infty} \mathrm{Re} \, \ln \left(1 - e^{2\pi i p \mathcal{T}} \right) ,
$$

where we have introduced $\mathcal{T} = \chi_{\bar{\alpha}_2} + i V_n^{-2}$. This term can be interpreted as due to world-sheet instantons, *i.e.* these are non-perturbative in $1/\alpha'$ [27] and first non-perturbative term for $s = 3/2$ can be interpreted as arising from (p, q)-strings [27].

7.3 $d = 7, E_4 = SL(5)$

Using equation (7.21) to convert equation (6.18) to string theory quantities we find the perturbative part of the $SL(5)$ automorphic function is given by

$$
g_d^{8s/5 - 2} \Phi_p = g_d^{-2} E_1 + g_d^{2s-3} V_n^{s-1/2} E_2 + g_d^{2s-3} V_n^{5/6 - s/3} \sum_{k=3}^{5} E_k P_k(\underline{\lambda}^1, \underline{\phi}) .
$$
(7.30)

Even though there are five terms there are only two different powers of g_d. For $s = (\Delta-1)/2$ then we find contributions at genus $g = 0, \Delta/2 - 1$ and so we find a physically acceptable perturbative series for any Δ. For the two cases of most interest, $s = 3/2$ and $s = 5/2$, Φ requires regularization. As discussed in section 4 for $s = 3/2$ the divergences arise in the third and fourth terms but they cancel and the result is a term with the same power of g_d and V_n but a logarithmic dependence on $\underline{\phi}$. For $s = 5/2$ the last term is divergent and can be subtracted off, leaving a term proportional to $\ln(g_d V_n^{-5/6})$. Thus the overall structure remains relatively unchanged, in particular one still finds contributions from two orders of perturbation theory. The non-perturbative part can be found from equation (3.21) using equation (7.21) to convert it to string variables.

7.4 $d = 6, E_5 = SO(5,5)$

Using equation (7.21) to convert equation (6.21) to string theory quantities we find the perturbative part of the $SO(5,5)$ automorphic function is given by

$$
\begin{aligned}
g_d^{2s-2}\Phi_p &= g_d^{-2}E_1 + g_d^{2s-3}V_n^{s-1/2}E_2 + g_d^{2s-3}V_n^{1/2}\sum_{k=3}^{8} E_k P_k(\underline{\lambda}^2, \underline{\phi}) \\
&+ g_d^{2s-3}V_n^{9/2-s}E_9 + g_d^{4s-12}E_{10} \ .
\end{aligned}
\tag{7.31}
$$

We observe that we have ten terms but only three different powers of g_d. Looking at the last terms we find it is a physically acceptable perturbative series if $s \geq \frac{5}{2}$ and for $s = (\Delta - 1)/2$ then we find contributions at genus $g = 0, \Delta/2 - 1, \Delta - 6$. There are divergences for $s \leq 5$ but as for $SL(5)$ these don't significantly affect the powers of g_d that appear, although for $s = 5$ the final term is replaced by a term proportional to $ln(g_d)$, so that only two orders of perturbation theory arise. Hence we find that for $s = \frac{3}{2}$ that the automorphic form is not relevant to string theory. In the next section we will consider a constrained SO(5,5) automorphic form, the ten dimensional vector using to construct it is taken to be null. This always has an acceptable perturbative series which we will discuss there.

7.5 $d = 5, E_6$

Using equation (7.21) to convert equation (6.21) to string theory quantities we find the perturbative part of the E_6 automorphic function is given by

$$g_d^{8s/3-2}\Phi_p = g_d^{-2}E_1 + g_d^{2s-3}V_n^{s-1/2}E_2 + g_d^{2s-3}V_n^{s/5+3/10}\sum_{k=3}^{12}E_kP_k(\underline{\lambda}^3,\underline{\phi})$$

$$+ g_d^{2s-3}V_n^{-3s/5+51/10}\sum_{k=13}^{17}E_kP_k(\underline{\lambda}^1,\underline{\phi}) + g_d^{4s-20}V_n^{2s/5-17/5}\sum_{k=18}^{22}E_kP_k(\underline{\lambda}^1,\underline{\phi})$$

$$+ g_d^{4s-20}V_n^{-2s/5+27/5}\sum_{k=23}^{27}E_kP_k(\underline{\lambda}^4,\underline{\phi})\ .$$

$$(7.32)$$

We have twenty seven terms but only three different powers of g_d. It is a physically acceptable perturbative series only if $s \geq \frac{9}{2}$. There are divergences if $s \leq 27/2$ but these don't significantly alter the powers of g_d that appear. If we take $s = (\Delta - 1)/2$ then we find contributions at genus $g = 0, \Delta/2 - 1, \Delta - 10$. We could consider an automorphic form that obeys an E_6-invariant cubic constraint and like the $SO(5,5)$ case this may well always have an acceptable perturbative series.

7.6 $d = 4, E_7$

Using equation (7.21) to convert equation (6.23) to string theory quantities we find the perturbative part of the E_7 automorphic function is given by

$$g_d^{4s-2}\Phi_p = g_d^{-2}E_1 + g_d^{2s-3}V_n^{s-1/2}E_2$$

$$+ g_d^{2s-3}V_n^{s/3+1/6}\sum_{k=3}^{17}E_kP_k(\underline{\lambda}^4,\underline{\phi}) + g_d^{2s-3}V_n^{-s/3+35/6}\sum_{k=18}^{32}E_kP_k(\underline{\lambda}^2,\underline{\phi})$$

$$+ g_d^{4s-35}V_n^{2s/3-61/6}\sum_{k=33}^{47}E_kP_k(\underline{\lambda}^2,\underline{\phi}) + g_d^{2s+12}V_n^{-s+29}E_{48}$$

$$+ g_d^{4s-36}V_n^{5}\sum_{k=49}^{68}E_kP_k(\underline{\lambda}^1 + \underline{\lambda}^5,\underline{\phi}) + g_d^{4s-36}V_n^{5}(E_{69} + E_{70})e^{-\frac{1}{\sqrt{2}}\sum_{\underline{\alpha}>0}\frac{\underline{\alpha}\cdot\underline{\phi}}{}}$$

$$+ g_d^{4s-36}V_n^{5}\sum_{k=71}^{85}E_kP_k(\underline{\lambda}^1 + \underline{\lambda}^5,\underline{\phi}) + g_d^{6s-121}V_n^{s-75/2}E_{86}$$

$$+ g_d^{4s-35}V_n^{s/3-53/6}\sum_{k=87}^{101}E_kP_k(\underline{\lambda}^4,\underline{\phi}) + g_d^{6s-136}V_n^{-2s/3+125/3}\sum_{k=102}^{116}E_kP_k(\underline{\lambda}^4,\underline{\phi})$$

$$+ g_d^{6s-136}V_n^{-s/3+67/3}\sum_{k=117}^{131}E_kP_k(\underline{\lambda}^2,\underline{\phi})$$

$$+ g_d^{6s-136}V_n^{1/2}E_{132} + g_d^{8s-268}V_n^{-s+133/2}E_{133}\ .$$

$$(7.33)$$

There are 133 terms but, in contrast to above, we find quite a few different powers of the coupling constant g_d. We observe that there is no value of s for which the series is acceptable as it will for any s involve odd and even powers of g_d. However we expect that the correct automorphic form is likely to be constructed by imposing a quartic E_7-invariant constraint on the lattice.

8. Perturbative Evaluation of the Constrained SO(5,5) Automorphic Form.

The vector representation of $SO(5,5)$ takes the from

$$|\psi> = \sum_{i=1}^{5} n^i |\vec{\mu}^i> + \sum_{i=6}^{10} m_i |\vec{\mu}^i> \tag{8.1}$$

Here the n^i and m_i belong to the $\bar{\mathbf{5}}$ and $\mathbf{5}$ representations of the $SL(5)$ subgroup. Since we are dealing with a vector it is $SO(5,5)$ invariant to impose that its length vanishes. This corresponds to the constraint

$$\sum_{i=1}^{5} n^i \tilde{m}_i = 0 \tag{8.2}$$

where $\tilde{m}_i = m_{11-i}$. As such rather than sum over the $SO(5,5)$ lattice as before we can sum subject to this constraint. Such a possibility was considered in reference [29] and we will use some of the technical tricks used there. In this section we will evaluate only the perturbative contribution of the corresponding automorphic form and so we can set all $\chi_{\vec{\alpha}} = 0$ from the outset. As such, we find that

$$|\varphi> = L(g^{-1})|\psi> = \sum_{i=1}^{5} n^i e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i} |\vec{\mu}^i> + \sum_{i=6}^{10} \tilde{m}_i e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\mu}^i} |\vec{\mu}^i> \ . \tag{8.3}$$

Taking $u = <\varphi|\varphi>$ we must evaluate

$$\Phi(\xi) = \sum_{\Lambda_c} \frac{1}{(u(\xi))^s} = \sum_{\Lambda} \frac{\pi^s}{\Gamma(s)} \int_0^1 d\theta \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi}{t}u} e^{2\pi i\theta \sum \tilde{m}_i n^i \tilde{m}_i} \ , \tag{8.4}$$

where Λ is the sum over all integers in the lattice; the above constraint being implemented by the integral over θ. We may write the sum over the ten integers as

$$\sum_{\Lambda} = \sum_{n^i, \tilde{m}_i=0}^{\wedge} + \sum_{n^i} \sum_{\tilde{m}_i}^{\wedge} \tag{8.5}$$

where the hat means that the term with all the integers vanish is excluded. The first sum leads to the expression

$$\Phi_1 \equiv \sum_{n^i}^{\wedge} \frac{1}{[(n^1)^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1} + \ldots + (n^5)^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^5}]^s} \quad . \tag{8.6}$$

This closely resembles the automorphic form for $SL(5)$ but the weights are not the same since they are the first five weights that occur in the ten representation of $SO(5,5)$. Thus the perturbative formula for Φ_1 simply consists of the first five terms in the perturbative part of the unconstrained $SO(5,5)$ automorphic form.

Let us denote the second term of equation (8.4) arising from the split in the sum given in equation (8.5) by Φ_2. It can be evaluated by apply the Poisson resummation formula to to the five integers n^i. This is possible as the sum of n^i is over all integers. Using equation (A.2), the result is

$$\Phi_2 = \sum_{\hat{n}_1} \sum_{m}^{\wedge} \frac{\pi^s}{\Gamma(s)} \int_0^1 d\theta \int_0^\infty \frac{dt}{t^{1+s-\frac{5}{2}}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}^1+\ldots+\vec{\mu}^5)}$$
$$e^{-\frac{\pi}{t}\sum_{i=6}^{10} \tilde{m}_i^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i}} e^{-\pi t \sum_{i=1}^5 (\hat{n}_i+\theta\tilde{m}_i)^2 e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i}} \tag{8.7}$$

We observe that $\hat{n}_i \to \hat{n}_i + \tilde{m}_i$ has the same effect as taking $\theta \to \theta + 1$. As such we may restrict the sum to $\hat{n}_i$ modulo $\tilde{m}_i$, but take the integral over θ to be from $-\infty$ to ∞. Completing the square on θ and changing to the variable y we find the expression becomes

$$\Phi_2 = \sum_{\hat{n} \bmod \tilde{m}} \sum_{\tilde{m}_i}^{\wedge} \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s-\frac{4}{2}}} \int_{-\infty}^\infty dy\, e^{-\pi y^2} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}^1+\ldots+\vec{\mu}^5)}$$
$$\times e^{-\frac{\pi}{t}\sum_{i=6}^{10} \tilde{m}_i^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i}} e^{-\pi t \left(-\frac{(\hat{n}\diamond\tilde{m})^2}{\tilde{m}\diamond\tilde{m}}+\hat{n}\diamond\hat{n}\right)} \frac{1}{\sqrt{\tilde{m}\diamond\tilde{m}}} \quad , \tag{8.8}$$

where

$$p \diamond q = \sum_{i=1}^5 p_i q_i e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i} \quad , \tag{8.9}$$

and

$$y = \left(\theta + \frac{\hat{n}\diamond\tilde{m}}{\tilde{m}\diamond\tilde{m}}\right) \sqrt{t\tilde{m}\diamond\tilde{m}} \quad . \tag{8.10}$$

We may carry out the integral over y which gives the factor 1. The expression of equation (8.8) can be broken into two terms depending if

$$(\hat{n}\diamond\tilde{m})^2 = (\hat{n}\diamond\hat{n})(\tilde{m}\diamond\tilde{m}) \tag{8.11}$$

or not. If it does not then we find a Bessel function which does not contain the perturbative terms we are trying to compute and so we discard this term. By the Schwarz inequality

equation (8.11) is only satisfied if $\hat{n}_i = \lambda \tilde{m}_i$ where λ is an integer such that this relation holds. The number of solutions for fixed $\tilde{m}_i$ is just the greatest common divisor (gcd) of $\tilde{m}_i$, *i.e.* gcd $(\tilde{m}_i)$. Thus our expression becomes

$$
\begin{aligned}
\Phi_2 &= \sum_m^{\wedge} \frac{\pi^s}{\Gamma(s)} \int_0^{\infty} \frac{dt}{t^{1+s-\frac{4}{2}}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}^1+...+\vec{\mu}^5)} \frac{e^{-\frac{\pi}{t}\sum_{i=6}^{10}\tilde{m}_i^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i}}}{\sqrt{\tilde{m}\cdot\tilde{m}}} gcd(\tilde{m}_i) \\
&= \sum_m^{\wedge} \frac{\pi^s}{\Gamma(s)} \frac{\Gamma(s-\frac{4}{2})}{\pi^{s-\frac{4}{2}}} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}^1+...+\vec{\mu}^5)} \frac{gcd(\tilde{m}_i)}{(\tilde{m}\diamond\tilde{m})^{s-\frac{5}{2}+1}}
\end{aligned}
\tag{8.12}
$$

In carrying out this step we have used that

$$
\tilde{m}\diamond\tilde{m} = \sum_{i=1}^5 \tilde{m}_i^2 e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i} = \sum_{i=6}^{10} \tilde{m}_i^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^i} \ ,
\tag{8.13}
$$

due to the fact that for the vector representation $\vec{\mu}^i = -\vec{\mu}^{11-i}$ and that $\tilde{m}_i \hat{=} m_{11-i}$.

To process this expression further we make use of the formulae

$$
\sum_r^{\wedge} \frac{1}{(r\cdot r)^l} = \sum_x^{\wedge} \frac{1}{x^{2l}} \sum_{r',coprime}^{\wedge} \frac{1}{(r'\cdot r')^l} = \zeta(2l) \sum_{r',coprime}^{\wedge} \frac{1}{(r'\cdot r')^l} \ ,
\tag{8.14}
$$

which arises from taking out the gcd x out of r. We use this formula to express the sum of $\tilde{m}_i$ in terms of coprimes and combine the gcd divisor which emerges together with the one already there and write the result in terms of a zeta function. We then use the formula again to rewrite the sum of coprimes in terms of an ordinary sum. The result of all this is that our expression is now given by

$$
\Phi_2 = \pi^2 \frac{\Gamma(s-\frac{4}{2})}{\Gamma(s)} \frac{\zeta(2s-4)}{\zeta(2s-3)} e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}^1+...+\vec{\mu}^5)} \sum_{\tilde{m}}^{\wedge} \frac{1}{(\tilde{m}\diamond\tilde{m})^{s-\frac{3}{2}}} \ .
\tag{8.15}
$$

Thus we find that the peturbative contribution to $SO(5,5)$ automorphic constructed using the vector representation is given by

$$
\begin{aligned}
\Phi(\xi) &= \Phi_1 + \Phi_2 \\
&= \sum_n^{\wedge} \frac{1}{[(n^1)^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1} + ... + (n^5)^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^5}]^s} \\
&\quad + \pi^2 \frac{\Gamma(s-\frac{4}{2})}{\Gamma(s)} \frac{\zeta(2s-4)}{\zeta(2s-3)} \sum_{\tilde{m}}^{\wedge} \frac{e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}^1+...+\vec{\mu}^5)}}{[\tilde{m}_1^2 e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1} + ... + \tilde{m}_5^2 e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}^5}]^{s-\frac{3}{2}}} \ .
\end{aligned}
\tag{8.16}
$$

We recognise the first term as having the same form as we found for $SL(5)$ although it is important to remember that the weights that occur here are those for the vector

representation of $SO(5,5)$, but only for the $\bar{\mathbf{5}}$ part in the decomposition to $SL(5)$. However we can still apply equation (3.19) to find that the result is given by

$$\Phi_1 = e^{-s\phi}e^{-\frac{s\rho}{\sqrt{2}x}}(E_1 + e^{(2s-1)\phi}E_2 + e^{s\phi}e^{\frac{(s-1)2x\rho}{\sqrt{2}}}\sum_{k=3}^{5}E_k P_k(\underline{\lambda}^2,\underline{\phi}))$$

$$= g_d^{2-2s}\left(g_d^{-2}E_1 + g_d^{(2s-3)}(V_n^{s-\frac{1}{2}}E_2 + V_n^{\frac{1}{2}}\sum_{k=3}^{5}E_k P_k(\underline{\lambda}^2,\underline{\phi}))\right) \ . \tag{8.17}$$

In the last line we have used the formula (7.21) relevant for $SO(5,5)$, that is taken $n = 4$ to convert from ϕ and ρ to the physical variables g_d and V_n.

We now evaluate the second term. Using the expression for the weights of the vector representation given in equation (6.3) we find that

$$e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}^1+\ldots+\vec{\mu}^5)} = e^{-\frac{\rho}{\sqrt{2}x}}e^{-\sqrt{2}\underline{\lambda}^1\cdot\phi} = V_n^{\frac{1}{2}}g_d^{-1}e^{-\sqrt{2}\underline{\lambda}^1\cdot\phi} \ . \tag{8.18}$$

Using equation (2.25) we may rewrite Φ_2 as

$$\Phi_2 = \pi^{2s-\frac{7}{2}}\frac{\Gamma(s-2)\Gamma(4-s)}{\Gamma(s)\Gamma(s-\frac{3}{2})}\frac{\zeta(2s-4)}{\zeta(2s-3)}\sum_m^{\wedge}\frac{1}{(m_1^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^1}+\ldots+\tilde{m}_5^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}^5})^{4-s}} \ . \tag{8.19}$$

We can use our previous formulae for $SL(5)$ to evaluate this term and then convert it to physical variables using $n = 4$ in equation (7.21) to find

$$\Phi_2 = \pi^{2s-\frac{7}{2}}\frac{\Gamma(s-2)\Gamma(4-s)}{\Gamma(s)\Gamma(s-\frac{3}{2})}\frac{\zeta(2s-4)}{\zeta(2s-3)}e^{(s-4)\phi}e^{\frac{(s-4)\rho}{\sqrt{2}x}}$$

$$\times\left(E_1' + e^{(-2s+7)\phi}E_2' + e^{(-s+4)\phi}e^{\frac{(-s+3)2x\rho}{\sqrt{2}}}\sum_{k=3}^{5}E_k' P_k(\underline{\lambda}^2,\underline{\phi})\right)$$

$$= \pi^{2s-\frac{7}{2}}\frac{\Gamma(s-2)\Gamma(4-s)}{\Gamma(s)\Gamma(s-\frac{3}{2})}\frac{\zeta(2s-4)}{\zeta(2s-3)}g_d^{2s-8}$$

$$\left(E_1' + g_d^{(-2s+7)}(V_n^{-s+\frac{7}{2}}E_2' + V_n^{-\frac{1}{2}}\sum_{k=3}^{5}E_k' P_k(\underline{\lambda}^2,\underline{\phi}))\right) \tag{8.20}$$

where $E_k' = 2\pi^{\frac{k-1}{2}}\zeta(-2s+9-k)\Gamma(-s+\frac{9}{2}-\frac{k}{2})/\Gamma(-s+4)$.

Let us now consider the case $s = \frac{3}{2}$. We observe that the Φ_2 part given in equation (8.15) contains an automorphic form at the value $s - \frac{3}{2} = 0$ which is equal to -1. The resulting contribution from Φ_2 is therefore

$$\frac{2}{3}\pi^2 e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}^1+\ldots+\vec{\mu}^5)} = \frac{2}{3}\pi^2 g_d^{-1}V_n^{\frac{1}{2}}e^{-\sqrt{2}\vec{\phi}\cdot\underline{\lambda}^1} \ . \tag{8.21}$$

One can find the same result from equation (8.20) where it is the last term that gives a non-zero result. Hence we find that for $s = \frac{3}{2}$ which corresponds to R^4 in seven dimensions we find that the result is given in Einstein frame by

$$\int d^6x \sqrt{-g} R^4 \Phi_{\frac{3}{2}} \, , \tag{8.22}$$

where the perturbative part is given by

$$\Phi_{p,\frac{3}{2}} = g_d^{-3}(E_1 + g_d^2(V_n E_2 + V_n^{\frac{1}{2}} \sum_{k=3}^{5} E_k P_k(\underline{\lambda}^2, \underline{\phi}))) + \frac{2}{3}\pi^2 g_d^{-1} V_n^{\frac{1}{2}} e^{-\sqrt{2}\vec{\phi}\cdot\vec{\lambda}^1} \, . \tag{8.23}$$

We observe that it only has six terms as opposed to the ten terms in the unconstrained automorphic form of equation (7.31). However the first five terms are just the same as the first five terms of this unconstrained automorphic form. To move to string frame one requires a factor of g_d and so we find the only contributions are at tree level and one loop. Hence we note, that unlike the unconstrained $SO(5,5)$ automorphic form, it gives an acceptable result.

Let us now consider the case of $s = \frac{5}{2}$. Examining equation (8.20) we see that the prefactors are finite except for a divergent numerator factor of $\zeta(1)$. However, there is also a $\zeta(1)$ divergent factor in the last of the first terms in Φ_1 which being at the end of the set of terms is not canceled by any term. Thus we find six terms which are divergent and examining them one finds that they are contained in an $SO(5,5)$ automorphic form for $s = \frac{3}{2}$. Indeed we may write

$$\Phi_{p,\frac{5}{2}} = \text{first four terms of } \Phi_1 + 4\zeta(1)\Phi_{p,\frac{3}{2}} \, . \tag{8.24}$$

However, we can regulate this in an $SO(5,5)$ manner by shifting $s \to s + \epsilon$, so that $\zeta(1) = \zeta(2s-4) \sim 1/2\epsilon$, and subtracting off and entire $SO(5,5)$ automorphic form $2\epsilon^{-1}\Phi_{3/2}$ (note that this also requires that the divergent non-perturbative part also cancels - as it must since regularization preserves the automorphic property). These correspond to a R^6 term in seven dimensions and and so we find that the effective action contains in Einstein frame the term

$$\int d^6x \sqrt{-g} R^6 \Phi_{\frac{5}{2}} \, . \tag{8.25}$$

where the perturbative part is given by

$$\Phi_{p,\frac{5}{2}} = g_d^{-5}(E_1 + g_d^4(V_n E_2 + V_n^{\frac{1}{2}} \sum_{k=3}^{4} E_k P_k(\underline{\lambda}^4, \underline{\phi}))) \, . \tag{8.26}$$

To move to string frame we require a factor of g_d^3 and so we find only a tree and two loop correction. This is in contrast to the unconstrained form of equation (7.31) which

possess ten terms all of which are acceptable from a perturbative viewpoint and indeed they contain the same powers of the coupling constant.

For $s \geq \frac{7}{2}$ we find that all the ten terms in $SO(5,5)$ constrained automorphic form are finite and they have an acceptable form when viewed from the string perspective. Indeed we find that in string frame the automorphic form corresponding to $D^{2\delta}R^{\frac{l_R}{2}}$, that is $s = \frac{\delta + l_R - 2}{4}$ has the following powers of the coupling

$$g_d^{-2}, \ g_d^{\frac{\delta + l_R - 8}{2}}, \ g_d^{\delta + l_R - 12} \ . \tag{8.27}$$

In fact the unconstrained automorphic form of equation (7.31) also has an acceptable coupling constant dependence which is the same as the constrained automorphic form. Furthermore the first five terms are the same. This strongly suggests that the perturbative parts of the constrained and unconstrained automorphic forms are the same. If so this would allows us to construct automorphic forms with no perturbative part by taking their difference.

Finally for the sake of completeness we note that we could have computed the "perturbative" part of the $SO(n,n)$ automorphic form in essentially the same way. In which case the analogue of equation (8.17) is given by

$$\Phi(\xi) = \sum_n^{\wedge} \frac{1}{(n_1^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}_1} + \ldots + n_n^2 e^{\sqrt{2}\vec{\phi}\cdot\vec{\mu}_n})^s}$$
$$+ \pi^{\frac{n}{2} - \frac{1}{2}} \frac{\Gamma(s - \frac{n}{2} + \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s - n + 1)}{\zeta(2s - n + 2)} \sum_m^{\wedge} \frac{e^{-\frac{1}{\sqrt{2}}\vec{\phi}\cdot(\vec{\mu}_1 + \ldots + \vec{\mu}_n)}}{(\tilde{m}_1^2 e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}_1} + \ldots + \tilde{m}_n^2 e^{-\sqrt{2}\vec{\phi}\cdot\vec{\mu}_n})^{s - \frac{n}{2} + 1}} \ , \tag{8.28}$$

and the formula (3.19) can be used to extract the perturbative part.

9. Discussion

In this paper we have used our previous method [31] to construct Eisenstein-like automorphic forms for any group and representation and found explicit formulae for the "perturbative" and "non-perturbative" parts in terms of the weights of the representation. Applying this to the groups E_{n+1} and the fundamental representation associated with node $n + 1$, we have explicitly computed the perturbative part in terms of the string coupling g_d in d dimensions and the volume of the torus V_n. We then examined the result to see if it could be physical, that is compatible with the result of a string theory calculation. In particular we looked for to see if it only contains terms of the form g_d^{2g-2} for g a positive integer. We have taken care to find expressions whose dependence on string quantities is very transparent. We note that the derivation is almost entirely group theoretic in nature involving the manipulation of properties of the representation and its decomposition into

representations of $SL(2) \otimes SL(n)$; the $SL(2)$ part being the well known symmetry of the IIB ten dimensional theory and the $SL(n)$ arising as the manifest symmetry of the n-torus.

For the case of dimensions $d \geq 7$, that is automorphic forms of $SL(N)$ groups based on the $\bar{\mathbf{N}}$ representation, we find that this is always the case and the perturbative series has only two terms. In six dimensions, we considered the automorphic forms for the group $SO(5,5)$ based on the vector, *i.e.* the $\mathbf{10}$ representation. The perturbation expansion is physical for $s \geq 5/2$, containing contributions from two ($s = 5/2$) or three ($s > 5/2$) orders of perturbation theory. However the $s = 3/2$ automorphic form that occurs with the R^4 term does not have a good perturbation expansion. To rectify this we also considered the constrained $SO(5,5)$ automorphic form based on a null vector representation. The resulting series are physical, containing contributions from two orders of perturbation theory, and we have computed them explicitly.

In five dimensions we considered the automorphic form for E_6 constructed from the $\mathbf{27}$ representation. The resulting perturbation expansions are physical if $s \geq 9/2$, where it only contains contributions from two ($s = 9/2$) or three ($s > 9/2$) orders in perturbation theory. Since the $\mathbf{27}$ representation of E_6 admits a cubic invariant there is a constrained automorphic form for E_6 which may have better agreement with string perturbation theory. In four dimensions we considered the automorphic form for E_7 constructed from the $\mathbf{133}$ representation and it appears to be unable to agree with string perturbation theory. In this case there is a quartic invariant of E_7 and therefore one can define a constrained E_7 automorphic form. The perturbative contributions from such constrained E_6 and E_7 automorphic forms is currently under investigation. In particular it is of interest to obtain automorphic forms which are consistent with the non-renormalization theorems of [34].

One important question is what representation should one take to construct the automorphic forms. In this paper we have taken the fundamental representations associated with the node $n + 1$ of the E_{n+1} Dynkin diagram (see figure 1). This is supported by the dimensional reduction calculation of [48], along the lines of reference [31], and outlined in the introduction, to identify the highest weight contained in the automorphic form. However it would also be of interest to evaluate the perturbative parts of automorphic forms based on other representations and see if they could be relevant to String Theory.

The automorphic forms we have considered are only convergent if $s > \frac{N}{2}$ where N is the dimensions of the representation. In ten dimensions, where $N = 2$, this condition is met for all terms of interest. However in compactified String Theory one readily finds that the automorphic forms that arise as coefficients for higher derivative terms are naively ill-defined at low orders. In this paper we used analytic continuation to define the automorphic forms for more general values of s. This still leaves some values of s where a regularization scheme is required due to poles in the complex s plane. In particular we chose to deform $s \rightarrow s + \epsilon$ and then subtract any poles in ϵ. Since this procedure preserves the automorphic

property of Φ one sees that in general the residue of any pole in ϵ must itself be an automorphic form. In the simplest examples, such as unconstrained automorphic forms with $s = N/2$, the residue is just a constant. However more generally there can be situations where the residue is itself a non-trivial automorphic form that needs to be subtracted off, as was the case for the constrained $SO(5,5)$ automorphic form with $s = 5/2$. Therefore one can expect there to be a rich interplay between regularization and automorphic forms that would be interesting to explore.

Non-holomorphic automorphic forms are non-analytic, unlike their better known cousins. However, their behaviour is partly controlled if they are required to obey Laplace type equations and, as advocated in [31], similar equations related to all the higher order Casimirs of the group from which they are constructed. The unconstrained automorphic forms for $SO(5,5)$ and E_{n+1} for $n = 5, 6.7$ are, following arguments given in [29], unlikely to obey such equations and one must adopt constraints in the sum over the integers to recover these equations. It would be interesting to investigate these equations more systematically for the automorphic forms considered in this paper. We also note that the non-Eisenstein automorphic forms found [23] in ten dimensions obey Laplace equations with sources. Clearly, there is much to be understood about non-holomorphic automorphic forms.

Note Added: It is instructive to compare our results with those of [37]. In the case of the fundamental representation of $SL(n)$ our automorphic forms agree with theirs and so do the corresponding perturbative parts. For $SO(5,5)$ these authors demonstrate that (their equation (3.54), in our notation)

$$\Phi^{SO(5,5)}_{s=3/2} = g_d^{-1}\left(\frac{2\zeta(3)}{g_d^2} + 2\Phi^{SO(4,4)}_{s=1}\right) . \tag{9.1}$$

If we use equations (3.19) and (8.28), applied to $SO(4,4)$, we find

$$\Phi^{SO(4,4)}_{p,s=1} = 2\zeta(2)e^{-\sqrt{2}\vec{\nu}^1\cdot\vec{\phi}} + 2\pi\zeta(1)e^{-\frac{1}{\sqrt{2}}(\vec{\nu}^2+\vec{\nu}^1)\cdot\vec{\phi}} - \pi\Gamma(0)e^{-\frac{1}{\sqrt{2}}(\vec{\nu}^2+\vec{\nu}^1)\cdot\vec{\phi}}$$
$$+ \frac{1}{3}\pi^2 e^{\frac{1}{\sqrt{2}}\vec{\nu}^4\cdot\vec{\phi}}e^{-\frac{1}{\sqrt{2}}(\vec{\nu}^3+\vec{\nu}^2+\vec{\nu}^1)\cdot\vec{\phi}} + \frac{1}{3}\pi^2 e^{-\frac{1}{\sqrt{2}}(\vec{\nu}^4+\vec{\nu}^3+\vec{\nu}^2+\vec{\nu}^1)\cdot\vec{\phi}} ,$$

(note that, strickly speaking, we should replace $s = 1$ with $s = 1 + \epsilon$ to obtain a finite answer). Here $\vec{\nu}^a$, $a = 1, 2, 3, 4$ are the first 4 weights of the 8-dimensional representation of $SO(4,4)$. The relevent weights to take are $\vec{\nu}^a = \vec{\mu}^{a+1}$, where $\vec{\mu}^i$, $i = 1, 2, 3, 4, 5$ are the first 5 weights of the 10-dimensional representation of $SO(5,5)$. Substituting this into (9.1) and using (7.21) to write $V_n = e^{-\sqrt{2}\vec{\nu}^1\cdot\vec{\phi}}$, one readily sees that the perturbative part of (9.1) is precisely (8.23).

Thus, for dimensions six and above, the perturbative parts of the automorphic forms in [37] agree with those found here. Therefore it is natural to expect that the complete

automorphic forms used in [37] are equal to the ones we have defined here by anayltic continuation. An exception is the seven dimensional case where an additional automorphic form constructed from the **10** of $SL(5)$ appears in [37], which was not considered in this paper. In addition we have proposed the $s = 5/2$ automorphic form defined below (8.24) for $SO(5,5)$, which was not considered in [37]. We showed that it has a good perurbative expansion and, like the $s = 3/2$ case, contains far fewer terms than those the occur at generic values of s.

Acknowledgements

Appendix: Formulae

Here we list some formulae that are used through the main text.

$$\frac{1}{u^s} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{1+s}} e^{-\frac{\pi u}{t}} \ . \tag{A.1}$$

Poisson resummation formula:

$$\sum_{\vec{m} \in \mathbf{Z}^N} e^{-\pi(\vec{m}-\vec{a}) \cdot A(\vec{m}-\vec{a}) + 2\pi i \vec{m} \cdot \vec{b}} = \sum_{\vec{\tilde{m}} \in \mathbf{Z}^N} \det A^{-\frac{1}{2}} e^{-\pi(\vec{\tilde{m}}+\vec{b}) \cdot A^{-1}(\vec{\tilde{m}}+\vec{b}) + 2\pi i (\vec{\tilde{m}}+\vec{b}) \cdot \vec{a}} \ . \tag{A.2}$$

Bessel function integral identity

$$\int_0^\infty \frac{dt}{t^{1+\lambda}} e^{-at-b/t} = 2 \left| \frac{a}{b} \right|^{\lambda/2} K_\lambda(2\sqrt{|ab|}) \ . \tag{A.3}$$

Asymptotic behaviour of Bessel function as $x \to \infty$

$$K_\lambda(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{l=0}^\infty \frac{1}{(2z)^l} \frac{\Gamma(\lambda+l+\frac{1}{2})}{\Gamma(l+1)\Gamma(\lambda-l+\frac{1}{2})} \ . \tag{A.4}$$

For two fundamental weights of $SL(n)$:

$$\underline{\lambda}^i \cdot \underline{\lambda}^j = \frac{i(n-j)}{n} \qquad i \le j \ . \tag{A.5}$$

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